Exercises to Relativistic Quantum Field Theory — Sheet 5
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Exercise 5.1 *Momentum of the quantised free scalar field* (1 point)

The energy P^0 and momentum $\vec{P}$ of a free real scalar field $\phi(x)$ are defined via the stress–energy tensor $T^{\mu\nu}$ as $P^\mu = \int \mathrm{d}^3x\, T^{\mu 0}$ and explicitly given by

$$P^0 = \int \mathrm{d}^3y\, \frac{1}{2}\left(\pi(y)^2 + (\nabla\phi(y))^2 + m^2\phi(y)^2\right), \qquad \vec{P} = -\int \mathrm{d}^3y\, \pi(y)(\nabla\phi(y)),$$

where $\pi(x) = \dot{\phi}(x)$ is the canonical conjugate operator to the field operator $\phi(x)$.

a) Using the canonical commutator relations, show that

$$[\phi(x), P^\mu] = \mathrm{i}(\partial^\mu\phi(x)).$$

b) Derive the following identity for a translation by a constant four-vector a^μ:

$$\exp(\mathrm{i}a_\mu P^\mu)\,\phi(x)\,\exp(-\mathrm{i}a_\nu P^\nu) = \phi(x+a).$$

Exercise 5.2 *2-particle phase space* (1 point)

We consider two particles with masses $m_{1,2}$ and four-momenta $p_{1,2}$ ($p_{1,2}^2 = m_{1,2}^2$). The total momentum is, thus, given by $k = p_1 + p_2$. The integral over the 2-particle phase space is defined as

$$\int \mathrm{d}\Phi_2 = \int \frac{\mathrm{d}^3p_1}{(2\pi)^3 2p_1^0} \int \frac{\mathrm{d}^3p_2}{(2\pi)^3 2p_2^0}\,(2\pi)^4\delta^{(4)}(k - p_1 - p_2)\bigg|_{p_{1,2}^0=\sqrt{m_{1,2}^2+\vec{p}_{1,2}^{\,2}}}.$$

a) Consider the decay of a particle of mass M and momentum k ($k^2 = M^2$) into two particles with momenta $p_{1,2}$. Show that the phase-space integral can be evaluated in the centre-of-mass frame as

$$\int \mathrm{d}\Phi_2 = \frac{1}{(2\pi)^2}\frac{\sqrt{\lambda(M^2, m_1^2, m_2^2)}}{8M^2}\,\theta(M - m_1 - m_2)\int \mathrm{d}\Omega_1,$$

where Ω_1 is the solid angle of particle 1, and $\lambda(x,y,z) = x^2+y^2+z^2-2xy-2xz-2yz$.

b) What is the phase-space integral of a $2 \to 2$ particle scattering reaction with incoming momenta $k_{1,2}$. What is the counterpart of M in this reaction?

Please turn over!

Exercise 5.3 *Charge operator of the free complex scalar field* (1 point)

Consider the field operator $\phi(x)$ of the free complex Klein-Gordon field describing a spin-0 boson of mass m and electric charge q. The plane-wave expansion of $\phi(x)$ is given by

$$\phi(x) = \int \mathrm{d}\tilde{p} \left(a(\vec{p})\mathrm{e}^{-\mathrm{i}px} + b^\dagger(\vec{p})\mathrm{e}^{\mathrm{i}px} \right),$$

where $a(\vec{p})$, $a^\dagger(\vec{p})$ are the annihilation and creation operators for the particle, respectively, and likewise $b(\vec{p})$, $b^\dagger(\vec{p})$ for the corresponding antiparticle. Express the charge operator

$$Q = \int \mathrm{d}^3x \ \mathrm{i}q \ {:}\left(\phi^\dagger(\partial_0\phi) - (\partial_0\phi)^\dagger\phi \right){:}$$

in terms of the annihilation and creation operators of the momentum eigenstates. The normal ordered form $:\mathcal{O}:$ of an operator $\mathcal{O}$ is defined in such a way that all annihilation operators a_i are to the right of all creation operators $a_j^\dagger$ (e.g. $:a_i a_j^\dagger: = :a_j^\dagger a_i: = a_j^\dagger a_i$).