Exercises to Relativistic Quantum Field Theory — Sheet 1
Prof. S. Dittmaier, Universität Freiburg, WS 2019/20

Exercise 1.1 *Some properties of Lorentz transformations* (2.5 points)

In the defining representation, Lorentz transformations comprise all 4×4 matrices Λ, transforming a four-vector $(a^\mu) = (a^0, \vec{a})$ to $a'^\mu = \Lambda^\mu{}_\nu a^\nu$, that leave the metric tensor $(g^{\mu\nu}) = \mathrm{diag}(+1,-1,-1,-1)$ invariant, i.e. $g^{\mu\nu} = \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta g^{\alpha\beta}$. In the following we consider the "proper orthochronous Lorentz group" $L_+^\uparrow$ that comprises all such Λ with the two constraints that $\det\Lambda = +1$ and $\Lambda^0{}_0 > 0$. The group $L_+^\uparrow$ consists of all rotations in space and all "boosts" which relate two frames of reference with a non-vanishing relative velocity.

a) A boost with relative velocity $\vec{\beta}$, is described by the matrix

$$(\Lambda(\vec{\beta})^\mu{}_\nu) = \begin{pmatrix} \gamma & -\gamma\vec{\beta}^{\mathrm{T}} \\ -\gamma\vec{\beta} & \mathbb{1} + (\gamma-1)\vec{e}\,\vec{e}^{\mathrm{T}} \end{pmatrix},$$

where $\vec{e}$ is defined by $\vec{\beta} = |\vec{\beta}|\vec{e}$, $|\vec{e}| = 1$, and $\gamma = (1-\vec{\beta}^2)^{-\frac{1}{2}}$. Calculate the boosted components x'^μ for the four-vectors $(x_\parallel^\mu) = (x^0, r\vec{e})$ and $(x_\perp^\mu) = (x^0, r\vec{e}_\perp)$ whose directions in space are parallel and perpendicular to the direction of $\vec{\beta}$, respectively, i.e. $\vec{e}_\perp \cdot \vec{e} = 0$, $|\vec{e}_\perp| = 1$.

b) Show that the sign of the time-like component a^0 of any non-space-like four-vector a^μ (i.e. $a^2 \geq 0$) is invariant under all Lorentz transformations $\Lambda \in L_+^\uparrow$.

c) Calculate $W = \Lambda(-\vec{\beta}_2)\Lambda(-\vec{\beta}_1)\Lambda(\vec{\beta}_2)\Lambda(\vec{\beta}_1)$ for small velocities $\vec{\beta}_1 = \beta_1\vec{e}_1$, $\vec{\beta}_2 = \beta_2\vec{e}_2$ and keep terms up to quadratic order in β_i, $i = 1, 2$. Here, $\vec{e}_i$ are the cartesian basis vectors in x^i direction. What kind of transformation is described by W?

d) Show that the totally antisymmetric tensor

$$\epsilon^{\mu\nu\rho\sigma} = \begin{cases} +1 & \text{if } (\mu\nu\rho\sigma) = \text{even permutation of } (0123), \\ -1 & \text{if } (\mu\nu\rho\sigma) = \text{odd permutation of } (0123), \\ 0 & \text{otherwise} \end{cases}$$

is an invariant tensor under all $\Lambda \in L_+^\uparrow$, i.e. $\epsilon'^{\mu\nu\rho\sigma} = \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta \Lambda^\rho{}_\gamma \Lambda^\sigma{}_\delta \epsilon^{\alpha\beta\gamma\delta} = \epsilon^{\mu\nu\rho\sigma}$.

e) Show that $d^\mu = \epsilon^{\mu\nu\rho\sigma} a_\nu b_\rho c_\sigma$ transforms like a four-vector under $\Lambda \in L_+^\uparrow$ if a^μ, b^μ and c^μ are four-vectors.

Please turn over!

Exercise 1.2 *Kinematics of a $1 \to 2$ particle decay* (2 points)

A particle of mass M and four-momentum k^μ decays into two particles of masses m_i and four-momenta p_i^μ $(i = 1, 2)$. The momenta obey their mass-shell conditions $k^2 = M^2$ and $p_i^2 = m_i^2$ and, in the centre-of-mass frame Σ, are given by

$$(k^\mu) = \begin{pmatrix} M \\ \vec{0} \end{pmatrix}, \qquad (p_i^\mu) = \begin{pmatrix} E_i \\ \vec{p}_i \end{pmatrix} \quad \text{with} \quad \vec{p}_i = |\vec{p}_i| \begin{pmatrix} \sin\theta_i \cos\phi_i \\ \sin\theta_i \sin\phi_i \\ \cos\theta_i \end{pmatrix}.$$

a) What are the consequences of four-momentum conservation $k = p_1 + p_2$ for the energies E_i, for the absolute values $|\vec{p}_i|$ of the three-momenta and for the angles θ_i and ϕ_i?

b) Calculate E_i and $|\vec{p}_i|$ as functions of the masses M and m_i.

c) The decaying particle is now considered in a frame Σ' in which the particle has the velocity β along the x^3 axis. What is the relation between energies and angles in Σ' with the respective quantities in Σ?

d) For the special case $m_1 = m_2 = 0$ (e.g. decay into two photons), determine the angle θ' between the directions of flight of the decay products in Σ' (i.e. the angle between $\vec{p}_1'$ and $\vec{p}_2'$) in terms of the parameters in Σ. What are the extremal values of θ'? In particular, discuss the cases $\beta = 0$ and $\beta \to 1$.