

Introduction to Relativistic Quantum Field Theory

Prof. Dr. Stefan Dittmaier

Albert-Ludwigs-Universität Freiburg, Physikalisches Institut
D-79104 Freiburg, Germany

Dr. Heidi Rzehak and Dr. Christian Schwinn are gratefully acknowledged for their contributions to this manuscript which was worked out during our common lectures on quantum field theory.

Summer Term 2018

Draft: July 9, 2018

Contents

1	Introduction	7
I	Quantization of Scalar Fields	9
2	Recapitulation of Special Relativity	11
2.1	Lorentz transformations, four-vectors, tensors	11
2.1.1	Minkowski space	11
2.1.2	Lorentz transformations	12
2.1.3	Differential operators	14
2.1.4	Tensors	15
2.2	Lorentz group and Lorentz algebra	15
2.2.1	Classification of Lorentz transformations	15
2.2.2	Infinitesimal transformations and group generators	17
2.3	Poincaré group and Poincaré algebra	19
2.3.1	The basics	19
2.3.2	Generators as differential operators	20
2.4	Relativistic point particles	22
3	The Klein–Gordon equation	25
3.1	Relativistic wave equation	25
3.2	Solutions of the Klein–Gordon equation	26
3.3	Conserved current	28
3.4	Interpretation of the Klein–Gordon equation	29
4	Classical Field Theory	31
4.1	Lagrangian and Hamiltonian formalism	31
4.1.1	Lagrangian field theory	31
4.1.2	Hamiltonian field theory	35
4.2	Actions for scalar fields	36
4.2.1	Free real scalar field	36
4.2.2	Free complex scalar field	37
4.3	Interacting fields	38

4.3.1	Scalar self-interactions	38
4.3.2	Explicit calculation of the Green function (<i>propagator</i>)	40
4.4	Symmetries and Noether Theorem	43
4.4.1	Continuous symmetries	43
4.4.2	Derivation of the Noether theorem	45
4.4.3	Internal symmetries and conserved currents	46
4.4.4	Translation invariance and energy-momentum tensor	47
5	Canonical quantization of free scalar fields	49
5.1	Canonical commutation relations	49
5.2	Free Klein–Gordon field	51
5.3	Particle states and Fock space	55
5.4	Field operator and wave function	58
5.5	Propagator and time ordering	59
6	Interacting scalar fields and scattering theory	61
6.1	Asymptotic states and S-matrix	61
6.2	Perturbation Theory	64
6.3	Feynman diagrams	68
6.3.1	Wick’s theorem	68
6.3.2	Feynman rules for the S-operator	69
6.3.3	Feynman rules for S-matrix elements	70
6.4	Cross sections and decay widths	76
II	Quantization of fermion fields	81
7	Representations of the Lorentz group	83
7.1	Lie groups and algebras	83
7.1.1	Definitions	83
7.1.2	Lie algebras	85
7.1.3	Irreducible representations	87
7.1.4	Constructing representations	89
7.2	Irreducible representations of the Lorentz group	90
7.3	Fundamental spinor representations	91
7.4	Product representations	93
7.5	Relativistic wave equations	95
7.5.1	Relativistic fields	95
7.5.2	Relativistic wave equations for free particles	96
7.5.3	The Dirac equation	97

8	Free Dirac fermions	99
8.1	Solutions of the classical Dirac equation	99
8.2	Quantization of free Dirac fields	102
8.2.1	Quantization procedure	102
8.2.2	Particle states and Fock space	103
8.2.3	Fermion propagator	107
8.2.4	Connection between spins and statistics	108
9	Interaction of scalar and fermion fields	109
9.1	Interacting fermion fields	109
9.2	Yukawa theory	111
9.2.1	Feynman rules for the S-operator	111
9.2.2	Feynman rules for S-matrix elements	113
III	Quantization of vector-boson fields	117
10	Free vector-boson fields	119
10.1	Classical Maxwell equations	119
10.2	Proca equation	121
10.3	Quantization of the elmg. field	122
10.3.1	Preliminaries	122
10.3.2	Gupta–Bleuler quantization:	122
10.4	Photon propagator	127
11	Interacting vector-boson fields	129
11.1	Electromagnetic interaction	129
11.2	Perturbation theory for spinor electrodynamics	132
11.2.1	Expansion of the S-operator	132
11.2.2	Feynman rules for S-matrix elements	134
11.3	Important processes of (spinor) QED	137
11.3.1	Elastic ep scattering	137
11.3.2	Other important processes in QED	140

Chapter 1

Introduction

Relativistic quantum field theory

= mathematical framework for description of elementary particles and their interactions

Guiding principles for the construction of field theory and of specific models of interactions:

- relativistic structure of space–time
- principles of quantum mechanics
- empirical knowledge collected in colliders experiments (mainly e^+e^- , $e^\pm p$, pp , $p\bar{p}$)

Some empirical facts on particle collisions:

- Particle creation and annihilation is possible in collisions.
- Relativistic kinematics (four-momentum conservation, conversion of mass and energy) is extremely well confirmed.
- The spectrum of observed particles is very rich, but only very few are really elementary:

– *Leptons* (spin 1/2): $e, \nu_e, \mu, \nu_\mu, \tau, \nu_\tau$

– *Quarks* (spin 1/2): $\underbrace{u, d, s, c, b}_{\text{confined in hadrons, i.e. mesons } (q\bar{q}) \text{ or baryons } (qqq)}, t$

confined in hadrons, i.e. mesons ($q\bar{q}$) or baryons (qqq)

– *Gauge bosons* (spin 1):

force carriers of the strong and electroweak interactions

gluons (confined) γ, Z^0, W^\pm bosons

– *Higgs boson* (spin 0):

lends mass to all elementary particles,
recently discovered (full identification ongoing)

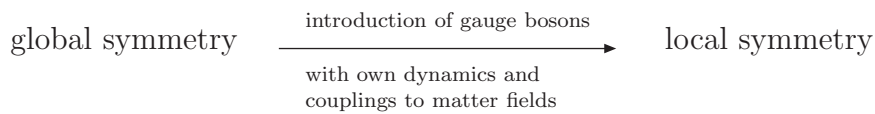
- Four fundamental interactions can be distinguished:
 - *electromagnetic interaction* } *electroweak* interaction
 - *weak interaction* }
 - *strong interaction*
 - *gravity* (not accessible by collider experiments)

Features of interaction models

- (special) relativistic covariance
 - description of particles by states in Hilbert space
 - description of particle dynamics by local fields
 - field quantization
- } relativistic quantum field theory
- internal symmetries between particles not only global, } gauge theories
 - but also local }

The role of symmetries

- space–time symmetry:
 - Lorentz/Poincaré covariant formulation of field theory
 - *mass* and *spin* as fundamental properties of particles
- internal symmetries:
 - unification of different particles into *multiplets* of symmetry groups
 - further quantum numbers (charge, isospin, etc.)
 - connection between symmetry and dynamics by *gauging* the symmetry:



The role of field quantization

- resolution of various inconsistencies in relativistic wave equations (negative-energy solutions, probability interpretation, etc.)
- wave-particle dualism
- creation and annihilation of particles
- connection between spin and statistics (bosons: $\text{spin} = 0, 1, \dots$; fermions: $\text{spin} = 1/2, 3/2, \dots$)

Part I

Quantization of Scalar Fields

Chapter 2

Recapitulation of Special Relativity

2.1 Lorentz transformations, four-vectors, tensors

2.1.1 Minkowski space

Definitions and notation:

- 3-vectors: $\vec{a} = (a^i)$, Latin indices: $i = 1, \dots, 3$
 \hookrightarrow span 3-dim. position space
- *Contravariant* 4-vectors: $a^\mu = (a^0, \vec{a})$, Greek indices: $\mu = 0, \dots, 3$
 \hookrightarrow span 4-dim. Minkowski space
- Space–time points (*events*): $x^\mu = (x^0, \vec{x}) = (ct, \vec{x})$
Natural units used in the following: $c \rightarrow 1, \hbar \rightarrow 1$
- *Metric tensor*: $(g_{\mu\nu}) = (g^{\mu\nu}) = \text{diag}(+1, -1, -1, -1)$

|| Comment:

Equations like $g^{\mu\nu} = g_{\mu\nu}$ —although correct for each coefficient—should be avoided, since the two sides correspond to two different geometrical objects.

- *Covariant* 4-vectors: $a_\mu = (a^0, -\vec{a}) = g_{\mu\nu}a^\nu$, $a^\mu = g^{\mu\nu}a_\nu$

Note: Einstein’s convention used, i.e. summation over pairs of equal upper and lower indices

- *Scalar product*:

$$a \cdot b = a^0 b^0 - \vec{a} \cdot \vec{b} = a^\mu b_\mu = a_\mu b^\mu = g_{\mu\nu} a^\mu b^\nu = g^{\mu\nu} a_\mu b_\nu \quad (2.1)$$

- *Length of 4-vectors*: $a^\mu a_\mu = (a^0)^2 - \vec{a}^2 = g_{\mu\nu} a^\mu a^\nu = \dots$

\hookrightarrow space–time distance s^2 of two events “ a ” and “ b ”:

$$s^2 = (x_a - x_b)^\mu (x_a - x_b)_\mu = (t_a - t_b)^2 - (\vec{x}_a - \vec{x}_b)^2 \quad (2.2)$$

Basic principles of special relativity

- *relativity principle* → laws of physics equivalent in all frames of inertia
- *constancy of speed of light* → value of c is equal in all frames

⇒ Scalar products (space–time distances, etc.) independent of frame of reference !

Classification of space–time distances: (independent of reference frame!)

$$(x_a - x_b)^2 = \left\{ \begin{array}{l} \text{time-like} \\ \text{light-like} \\ \text{space-like} \end{array} \right\} \quad \text{if } (x_a - x_b)^2 \left\{ \begin{array}{l} > 0 \\ = 0 \\ < 0 \end{array} \right\}, \quad \text{i.e. } |t_a - t_b| - |\vec{x}_a - \vec{x}_b| \left\{ \begin{array}{l} > 0 \\ = 0 \\ < 0 \end{array} \right. \quad (2.3)$$

- Time-like: Signals with velocity $<$ speed of light c can be sent from x_a^μ to x_b^μ .
- Light-like: If $t_b > t_a$, a light ray can be sent from x_a^μ to x_b^μ .
- Space-like: There is a frame with $t_a = t_b$, i.e. “ a ” and “ b ” happen simultaneously.

All events “ b ” with $(x_a - x_b)^2 = 0$ form the *light-cone* of x_a .

⇒ The light cone of x_a separates events causally connected/disconnected to “ a ”.

2.1.2 Lorentz transformations

= all coordinate transformations of Minkowski space that leave the space–time distances (2.2) invariant

Definitions:

- *Homogenous Lorentz transformations* = all *linear* transformations characterized by 4×4 matrix Λ :

$$a'^\mu = \Lambda^\mu{}_\nu a^\nu, \quad \text{matrix notation (contravariant vectors!): } a' = \Lambda a \quad (2.4)$$

obeying the invariance property $a'^2 = a^2$ for all 4-vectors a^μ

⇒ All scalar products are invariant:

$$a' \cdot b' = \frac{1}{2}(a' + b')^2 - \frac{1}{2}a'^2 - \frac{1}{2}b'^2 = \frac{1}{2}(a + b)^2 - \frac{1}{2}a^2 - \frac{1}{2}b^2 = a \cdot b \quad (2.5)$$

⇒ Invariance property of Λ :

$$g_{\mu\nu} a'^\mu b'^\nu = g_{\mu\nu} \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma a^\rho b^\sigma \stackrel{!}{=} g_{\rho\sigma} a^\rho b^\sigma \quad \Rightarrow \quad g_{\mu\nu} \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma \stackrel{!}{=} g_{\rho\sigma}, \quad \Lambda^T g \Lambda = g \quad (2.6)$$

- *Inhomogenous Lorentz transformations (Poincaré transformations)*

= all *affine* transformations of space–time

characterized by 4×4 matrix Λ and 4-vector a :

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu, \quad x' = \Lambda x + a \quad (2.7)$$

⇒ At least all space–time distances invariant: $(x'_a - x'_b)^2 = (x_a - x_b)^2$

Examples:

- Rotations:

$$\Lambda_D = \begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix} \quad \text{with } D^T D = 1, \quad \text{so that } \Lambda_D^T g \Lambda_D = g \quad (2.8)$$

Rotation around the x^3 axis:

$$D_3(\varphi) = \begin{pmatrix} \cos \varphi & -\sin \varphi & 0 \\ \sin \varphi & \cos \varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (2.9)$$

- *Boosts* (relating inertial frames moving with a relative velocity \vec{v}):

Boost B_3 in the x^3 direction:

$$\begin{aligned} t' &= \gamma(t + v x^3), & \gamma &= 1/\sqrt{1 - v^2} \\ x'^1 &= x^1, \\ x'^2 &= x^2, \\ x'^3 &= \gamma(x^3 + vt) \end{aligned} \quad (2.10)$$

Convenient parametrization of Λ_{B_3} by *rapidity* ν , where $v = \tanh \nu$:

$$\Lambda_{B_3} = \begin{pmatrix} \gamma & 0 & 0 & \gamma v \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma v & 0 & 0 & \gamma \end{pmatrix} = \begin{pmatrix} \cosh \nu & 0 & 0 & \sinh \nu \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh \nu & 0 & 0 & \cosh \nu \end{pmatrix} = \Lambda_{B_3}(\nu) \quad (2.11)$$

Comment:

The angles $\varphi \vec{e}$ ($0 \leq \varphi \leq \pi$, $|\vec{e}| = 1$) that parametrize all rotations define a compact set (the angles $\pm \pi \vec{e}$ correspond to the same rotation).

The rapidities $\nu \vec{e}$ are contained in a non-compact set of numbers ($-\infty < \nu < \infty$).

⇒ The Lorentz transformations form a *non-compact Lie group*. See also below.

Inverse Lorentz transformations

$$a'^{\mu} = \Lambda^{\mu}_{\nu} a^{\nu}, \quad \text{i.e.} \quad a^{\mu} = (\Lambda^{-1})^{\mu}_{\nu} a'^{\nu}$$

Proposition:

$$(\Lambda^{-1})^{\mu}_{\nu} = g_{\nu\alpha} \Lambda^{\alpha}_{\beta} g^{\beta\mu} \equiv \Lambda_{\nu}^{\mu} \quad (2.12)$$

Proof:

Verify $\Lambda^{-1} \Lambda = \mathbf{1}$ ($\Lambda \Lambda^{-1} = \mathbf{1}$ analogously):

$$\Lambda_{\nu}^{\mu} \Lambda^{\nu}_{\rho} = g_{\nu\alpha} \Lambda^{\alpha}_{\beta} g^{\beta\mu} \Lambda^{\nu}_{\rho} = (g_{\nu\alpha} \Lambda^{\alpha}_{\beta} \Lambda^{\nu}_{\rho}) g^{\beta\mu} \stackrel{(2.6)}{=} g_{\beta\rho} g^{\beta\mu} = g^{\mu}_{\rho} = \delta^{\mu}_{\rho} \quad (2.13)$$

q.e.d.

Application: Lorentz transformation of covariant 4-vectors

$$x'_{\mu} = g_{\mu\nu} x'^{\nu} = g_{\mu\nu} \Lambda^{\nu}_{\rho} x^{\rho} = g_{\mu\nu} \Lambda^{\nu}_{\rho} g^{\rho\sigma} x_{\sigma} = \Lambda_{\mu}^{\sigma} x_{\sigma}. \quad (2.14)$$

2.1.3 Differential operators

Definitions:

- *covariant 4-gradient:*

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial t}, \vec{\nabla} \right) \quad (2.15)$$

- *contravariant 4-gradient:*

$$\partial^\mu \equiv \frac{\partial}{\partial x_\mu} = g^{\mu\nu} \partial_\nu = \left(\frac{\partial}{\partial t}, -\vec{\nabla} \right) \quad (2.16)$$

- *wave (d'Alembert) operator:*

$$\square \equiv \partial_\mu \partial^\mu = \frac{\partial^2}{\partial t^2} - \vec{\nabla}^2 \quad (2.17)$$

Lorentz/Poincaré transformation properties: $x'^\mu = \Lambda^\mu_\nu x^\nu + a^\mu$

- 4-gradients:

$$\partial'_\mu = \frac{\partial}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu} \frac{\partial}{\partial x^\nu} = \Lambda_\mu^\nu \partial_\nu, \quad \partial'^\mu = \dots = \Lambda^\mu_\nu \partial^\nu \quad (2.18)$$

- wave operator:

$$\square' = \partial'_\mu \partial'^\mu = \Lambda_\mu^\nu \Lambda^\mu_\rho \partial_\nu \partial^\rho = g^\nu_\rho \partial_\nu \partial^\rho = \partial_\nu \partial^\nu = \square = \text{invariant} \quad (2.19)$$

- *4-divergence* of a vector field V^μ :

$$\partial_\mu V^\mu(x) = \partial_0 V^0(x) + \partial_i V^i(x) = \dot{V}^0(x) + \vec{\nabla} \cdot \vec{V} = \text{invariant} \quad (2.20)$$

2.1.4 Tensors

Definitions

- *Contravariant tensor* $T^{\mu_1 \dots \mu_n}$ of rank n = object that transforms like the direct product $a^{\mu_1} \dots a^{\mu_n}$ under the change of coordinate frames, i.e.

$$T^{\mu_1 \dots \mu_n} = \Lambda^{\mu_1}_{\rho_1} \dots \Lambda^{\mu_n}_{\rho_n} T^{\rho_1 \dots \rho_n}. \quad (2.21)$$

- A *covariant tensor* $T_{\mu_1 \dots \mu_n}$ of rank n transforms like $a_{\mu_1} \dots a_{\mu_n}$.
- A *mixed-rank* (n, m) *tensor* transforms as

$$T^{\mu_1 \dots \mu_n}_{\nu_1 \dots \nu_m} = \Lambda^{\mu_1}_{\rho_1} \dots \Lambda^{\mu_n}_{\rho_n} \Lambda_{\nu_1}^{\sigma_1} \dots \Lambda_{\nu_m}^{\sigma_m} T^{\rho_1 \dots \rho_n}_{\sigma_1 \dots \sigma_m}. \quad (2.22)$$

Invariant tensors:

- Metric tensor: $g^{\mu\nu} = g^{\nu\mu}, \quad g'_{\mu\nu} = g_{\mu\nu}$
- Totally antisymmetric tensor:

$$\epsilon^{\mu\nu\rho\sigma} = \begin{cases} +1 & \text{if } (\mu\nu\rho\sigma) = \text{even permutation of } (0123) \\ -1 & \text{if } (\mu\nu\rho\sigma) = \text{odd permutation of } (0123) \\ 0 & \text{otherwise} \end{cases} \quad (2.23)$$

Transformation:

$$\epsilon'^{\mu\nu\rho\sigma} = \epsilon^{\mu\nu\rho\sigma} \times \det \Lambda = \pm \epsilon^{\mu\nu\rho\sigma} = \text{invariant pseudo-tensor} \quad (2.24)$$

(see Exercise 1.1)

2.2 Lorentz group and Lorentz algebra

2.2.1 Classification of Lorentz transformations

Definition: The set of Lorentz transformations forms the *Lorentz group* L :

- closure: $\Lambda_1 \Lambda_2 = \Lambda = \text{Lorentz transformation}$ (prove!)
- associativity: $\Lambda_1 (\Lambda_2 \Lambda_3) = (\Lambda_1 \Lambda_2) \Lambda_3$
- unit element: $(\Lambda_e)^\mu_{\nu} = \delta^\mu_{\nu}$
- inverse elements: $(\Lambda^{-1})^\mu_{\nu} = \Lambda_{\nu}^{\mu}$

Important discrete Lorentz transformations:

- *parity* P (space inversion):

$$x^\mu \rightarrow x'^\mu = \Lambda_P{}^\mu{}_\nu x^\nu = \begin{pmatrix} x^0 \\ -\vec{x} \end{pmatrix}, \quad \Lambda_P = \text{diag}(+1, -1, -1, -1) \quad (2.25)$$

- time reversal T :

$$x^\mu \rightarrow x'^\mu = \Lambda_T{}^\mu{}_\nu x^\nu = \begin{pmatrix} -x^0 \\ \vec{x} \end{pmatrix}, \quad \Lambda_T = \text{diag}(-1, +1, +1, +1) \quad (2.26)$$

Invariant properties of Λ matrices and classification:

- $\det \Lambda = \pm 1$, since $\Lambda^T g \Lambda = g \quad \Rightarrow \text{Def.: } L_\pm \equiv \{\Lambda \mid \det \Lambda = \pm 1\}$
 L_+ = subgroup of *proper* Lorentz transformations ($L_- \neq$ subgroup)
- $|\Lambda^0_0| \geq 1$, since $g_{00} = 1 = g_{\mu\nu} \Lambda^\mu_0 \Lambda^\nu_0 = (\Lambda^0_0)^2 - (\Lambda^i_0)^2$
 Def.: $L^\uparrow \equiv \{\Lambda \mid \Lambda^0_0 \geq 1\}$, $L^\downarrow \equiv \{\Lambda \mid \Lambda^0_0 \leq -1\}$
 L^\uparrow = subgroup of *orthochronous* Lorentz transformations ($L^\downarrow \neq$ subgroup)
- Consequence: break-up of the Lorentz group into four disconnected subsets

	$\det \Lambda :$	$\Lambda^0_0 :$	Example:
L_+^\uparrow	+	> 1	$\Lambda = \mathbf{1}$
L_-^\uparrow	-1	> 1	$\Lambda = \Lambda_P$
L_-^\downarrow	-1	< -1	$\Lambda = \Lambda_T$
L_+^\downarrow	+1	< -1	$\Lambda = -\mathbf{1} = \Lambda_P \Lambda_T$

(2.27)

Def.: $L_+^\uparrow \equiv \{\Lambda \mid \det \Lambda = +1, \Lambda^0_0 \geq 1\}$
 = group of *proper, orthochronous (special)* Lorentz transformations

- Decomposition of Λ (non-trivial!):
 Each $\Lambda \in L_+^\uparrow$ can be written as a product of a rotation and a boost:

$$\Lambda = \Lambda_B \Lambda_D. \quad (2.28)$$

The rotations form a subgroup of L_+^\uparrow , while the boosts do not.

2.2.2 Infinitesimal transformations and group generators

Infinitesimal rotations and boosts:

Consider the rotation (2.9) and the boost (2.11) for infinitesimal parameters $\delta\varphi$ and $\delta\nu$:

$$\Lambda_{D_3}(\delta\varphi) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\delta\varphi & 0 \\ 0 & \delta\varphi & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \mathcal{O}(\delta\varphi^2) \equiv \mathbf{1} - i\delta\varphi J^3 + \mathcal{O}(\delta\varphi^2), \quad (2.29)$$

$$\Lambda_{B_3}(\delta\nu) = \begin{pmatrix} 1 & 0 & 0 & \delta\nu \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \delta\nu & 0 & 0 & 1 \end{pmatrix} + \mathcal{O}(\delta\nu^2) \equiv \mathbf{1} - i\delta\nu K^3 + \mathcal{O}(\delta\nu^2), \quad (2.30)$$

General infinitesimal rotations or boosts parametrized by six parameters $\delta\varphi_i$ and $\delta\nu_i$:

$$\Lambda_D(\delta\varphi_i) = \mathbf{1} - i\delta\vec{\varphi} \cdot \vec{J} = \mathbf{1} - i\delta\varphi_i J^i, \quad \Lambda_B(\delta\nu_i) = \mathbf{1} - i\delta\vec{\nu} \cdot \vec{K} = \mathbf{1} - i\delta\nu_i K^i \quad (2.31)$$

Definitions:

J^i = generator of infinitesimal rotations around the x^i axis (*angular momentum*)

K^i = generator of infinitesimal boosts in the x^i direction

Properties of the generators:

- Explicitly:

$$J^1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}, \quad J^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, \quad J^3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (2.32)$$

$$K^1 = \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad K^2 = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad K^3 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} \quad (2.33)$$

- Hermiticity:

$$J^{i\dagger} = J^i = \text{hermitian}, \quad K^{i\dagger} = -K^i = \text{anti-hermitian} \quad (2.34)$$

- Commutation relations:

$$[J^i, J^j] = i\epsilon^{ijk} J^k \quad (\text{relations of angular momentum}) \quad (2.35)$$

$$[J^i, K^j] = i\epsilon^{ijk} K^k \quad (\vec{K} \text{ transforms as 3-vector operator}) \quad (2.36)$$

$$[K^i, K^j] = -i\epsilon^{ijk} J^k. \quad (2.37)$$

Comment:

The third equation expresses the fact that boosts do not form a subgroup of L , but that the product of two boosts in general involves a rotation (the so-called *Wigner rotation*).

General infinitesimal Lorentz transformations:

- General form: $\Lambda^\mu{}_\nu(\delta\omega) = \delta^\mu_\nu + \delta\omega^\mu{}_\nu + \dots$

\Leftrightarrow condition for Lorentz transformation:

$$\begin{aligned} g_{\mu\nu}\Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma &= g_{\mu\nu}(\delta^\mu_\rho + \delta\omega^\mu{}_\rho)(\delta^\nu_\sigma + \delta\omega^\nu{}_\sigma) + \dots \\ &= g_{\rho\sigma} + \delta\omega_{\sigma\rho} + \delta\omega_{\rho\sigma} + \dots \stackrel{!}{=} g_{\rho\sigma}, \end{aligned} \quad (2.38)$$

$\Rightarrow \delta\omega$ are antisymmetric,

$$\delta\omega_{\sigma\rho} = -\delta\omega_{\rho\sigma}, \quad (2.39)$$

and comprise six independent entries corresponding to $\delta\varphi_i$ and $\delta\nu_i$.

- Generators $M^{\alpha\beta}$:

$$\Lambda^\mu{}_\nu(\delta\omega) = \delta^\mu_\nu + \delta\omega_{\alpha\beta} g^{\alpha\mu} \delta^\beta_\nu \equiv \delta^\mu_\nu - \frac{i}{2} \delta\omega_{\alpha\beta} (M^{\alpha\beta})^\mu{}_\nu \quad (2.40)$$

$$\Rightarrow (M^{\alpha\beta})^\mu{}_\nu = i(g^{\alpha\mu} \delta^\beta_\nu - g^{\beta\mu} \delta^\alpha_\nu) \quad (2.41)$$

Matrix notation: $\Lambda(\delta\omega) = \mathbf{1} - \frac{i}{2} \delta\omega_{\alpha\beta} M^{\alpha\beta}$

- Connection between J^i , K^i and $M^{\alpha\beta}$:

$$K^j = M^{0j}, \quad (K^j)^\mu{}_\nu = i(g^{0\mu} \delta^j_\nu - g^{j\mu} \delta^0_\nu) = \begin{cases} i, & (\mu, \nu) = (0, j) \text{ or } (j, 0), \\ 0 & \text{otherwise,} \end{cases} \quad (2.42)$$

$$\begin{aligned} J^k &= \frac{1}{2} \epsilon^{ijk} M^{ij}, \quad (J^k)^{mn} = \frac{i}{2} \epsilon^{ijk} (g^{im} \delta_{jn} - g^{jm} \delta_{in}) = -i\epsilon^{mnk} \\ &\text{(e.g. } J^3 = M^{12}\text{)} \end{aligned} \quad (2.43)$$

- Commutators: (*Lorentz algebra*)

$$[M^{\mu\nu}, M^{\rho\sigma}] = -i(g^{\mu\rho} M^{\nu\sigma} - g^{\mu\sigma} M^{\nu\rho} - g^{\nu\rho} M^{\mu\sigma} + g^{\nu\sigma} M^{\mu\rho}) \quad (2.44)$$

|| Comment:
|| Proof straightforward, easiest based on commutators for J^i , K^i .

Finite Lorentz transformations:

\Leftrightarrow result from infinitesimal transformations upon (matrix) exponentiation:

$$\Lambda_D(\vec{\varphi}) = \exp(-i\varphi_i J^i), \quad \Lambda_B(\vec{\nu}) = \exp(-i\nu_i K^i), \quad \Lambda(\omega) = \exp\left(-\frac{i}{2} \omega_{\alpha\beta} M^{\alpha\beta}\right) \quad (2.45)$$

|| Comment:
|| The iterative limit $\Lambda_D(\varphi_i) = \lim_{N \rightarrow \infty} \left(\mathbf{1} - i\frac{\varphi_i}{N} J^i\right)^N$ is demonstrative, but not of much practical use.

2.3 Poincaré group and Poincaré algebra

2.3.1 The basics

Definition: The *Poincaré group* P is the group of all inhomogeneous Lorentz transformations (Λ, a) with $x' = \Lambda x + a$, where $\Lambda \in L$, $a =$ any 4-vector. Obvious restrictions are P_+^\uparrow with $\Lambda \in L_+^\uparrow$, etc.

$\Rightarrow P, P_+^\uparrow$, etc. are non-compact Lie groups with 10 independent parameters.

Subgroups:

- $(\Lambda, 0)$: groups L, L_+^\uparrow , etc.
- $(\mathbf{1}, a)$: (abelian) group T_4 of 4-dim. translations

Composition law:

$$(\Lambda_2, a_2) (\Lambda_1, a_1) = \underbrace{(\Lambda_2 \Lambda_1, \Lambda_2 a_1 + a_2)}_{\text{as in } L} \quad (2.46)$$

|| Comment:

|| Proof: Insert $x' = \Lambda_1 x + a_1$ into $x'' = \Lambda_2 x' + a_2$ to obtain $x'' = \Lambda_2 (\Lambda_1 x + a_1) + a_2 = \Lambda_2 \Lambda_1 x + \Lambda_2 a_1 + a_2$.

$\Rightarrow P$ is *semi-direct* product: $P = T_4 \rtimes L$

Infinitesimal transformations and generators:

- General transformation:

$$(\Lambda, a) = \exp \left(-\frac{i}{2} \omega_{\alpha\beta} M^{\alpha\beta} + i a_\mu \underbrace{P^\mu}_{\text{generators for translations (4-momentum)}} \right) \quad (2.47)$$

- Infinitesimal transformation:

$$(\mathbf{1} + \delta\omega, \delta a) = \mathbf{1} - \frac{i}{2} \delta\omega_{\alpha\beta} M^{\alpha\beta} + i \delta a_\mu P^\mu + \dots \quad (2.48)$$

- Poincaré algebra:

$$[M^{\mu\nu}, M^{\rho\sigma}] = \dots, \quad \text{as in } L_+^\uparrow, \quad (2.49)$$

$$[P^\mu, P^\nu] = 0, \quad \text{since } T_4 \text{ abelian}, \quad (2.50)$$

$$[P^\mu, M^{\rho\sigma}] = i(g^{\mu\rho} P^\sigma - g^{\mu\sigma} P^\rho) \quad (2.51)$$

Proof of (2.51):

$$\begin{aligned}
(\mathbf{1} + \delta\omega, 0)^{-1} (\mathbf{1}, \delta a) (\mathbf{1} + \delta\omega, 0) &= \left(\mathbf{1} + \frac{i}{2} \delta\omega_{\alpha\beta} M^{\alpha\beta} \right) (\mathbf{1} + i\delta a_{\mu} P^{\mu}) \left(\mathbf{1} - \frac{i}{2} \delta\omega_{\alpha\beta} M^{\alpha\beta} \right) + \dots \\
&= \mathbf{1} + i\delta a_{\mu} P^{\mu} - \frac{1}{2} \delta\omega_{\alpha\beta} \delta a_{\mu} [M^{\alpha\beta}, P^{\mu}] + \dots \\
(\mathbf{1} + \delta\omega, 0)^{-1} (\mathbf{1}, \delta a) (\mathbf{1} + \delta\omega, 0) &= (\mathbf{1} - \delta\omega, 0) (\mathbf{1} + \delta\omega, \delta a) \\
&= (\mathbf{1}, \delta a^{\mu} - \delta\omega^{\mu\nu} \delta a_{\nu}) \\
&= \mathbf{1} + i(\delta a^{\mu} - \delta\omega^{\mu\nu} \delta a_{\nu}) P_{\mu} + \dots \\
&= \mathbf{1} + i\delta a_{\mu} P^{\mu} - \frac{i}{2} \delta\omega_{\mu\nu} (\delta a^{\nu} P^{\mu} - \delta a^{\mu} P^{\nu}) + \dots
\end{aligned}$$

\Leftrightarrow (2.51) follows upon comparing coefficients for arbitrary $\delta\omega_{\alpha\beta} \delta a_{\mu}$.

q.e.d.

2.3.2 Generators as differential operators

Inspect operation of transformations (Λ, a) on some scalar function $\phi(x)$:

$$\phi(x) \xrightarrow{(\Lambda, a)} \phi'(x') = \phi'(\Lambda x + a) \stackrel{!}{=} \phi(x), \quad \text{i.e.} \quad \phi'(x) = \phi(\Lambda^{-1}(x - a)) \quad (2.52)$$

- Translations:

- infinitesimal:

$$\begin{aligned}
\phi'(x) &= \phi(x - \delta a) = \phi(x) - \delta a^{\mu} \partial_{\mu} \phi(x) + \dots \\
&= [1 + i\delta a^{\mu} (i\partial_{\mu}) + \dots] \phi(x) \\
&\equiv [1 + i\delta a^{\mu} P_{\mu} + \dots] \phi(x), \quad (2.53)
\end{aligned}$$

$\Rightarrow P^{\mu} = i\partial^{\mu} = (i\partial_t, -i\vec{\nabla}) = 4\text{-momentum operator as differential operator}$

- finite transformations:

$$\phi'(x) = \phi(x - a) = \exp\{ia_{\mu} P^{\mu}\} \phi(x) \quad (2.54)$$

- Homogeneous Lorentz transformations:

- infinitesimal:

$$\begin{aligned}
\phi'(x) &= \phi(\Lambda^{-1}x) = \phi\left(x^{\mu} + \frac{i}{2} \delta\omega_{\alpha\beta} (M^{\alpha\beta})^{\mu}_{\nu} x^{\nu} + \dots\right) \\
&= \phi(x) + \frac{i}{2} \delta\omega_{\alpha\beta} \underbrace{(M^{\alpha\beta})^{\mu}_{\nu}}_{= i(g^{\alpha\mu} \delta_{\nu}^{\beta} - g^{\beta\mu} \delta_{\nu}^{\alpha})} x^{\nu} \partial_{\mu} \phi(x) + \dots \\
&= \phi(x) + \frac{i}{2} \delta\omega_{\alpha\beta} i(x^{\beta} \partial^{\alpha} - x^{\alpha} \partial^{\beta}) \phi(x) + \dots \\
&\equiv \phi(x) - \frac{i}{2} \delta\omega_{\alpha\beta} L^{\alpha\beta} \phi(x) + \dots \quad (2.55)
\end{aligned}$$

$\Rightarrow L^{\alpha\beta} = i(x^{\alpha} \partial^{\beta} - x^{\beta} \partial^{\alpha}) = x^{\alpha} P^{\beta} - x^{\beta} P^{\alpha}$

= generalized angular momentum operator as differential operator

– finite transformations:

$$\phi'(x) = \phi(\Lambda^{-1}x) = \exp\left\{-\frac{i}{2}\omega_{\alpha\beta}L^{\alpha\beta}\right\}\phi(x) \quad (2.56)$$

• General case:

$$\phi'(x) = \phi(\Lambda^{-1}(x - a)) = \exp\left\{ia_{\mu}P^{\mu} - \frac{i}{2}\omega_{\alpha\beta}L^{\alpha\beta}\right\}\phi(x) \quad (2.57)$$

2.4 Relativistic point particles

Point particle in non-relativistic mechanics: momentum = $m\dot{\vec{x}}$

But: $\left(\frac{dx^\mu}{dt}\right) = (1, \dot{\vec{x}}) \neq$ 4-vector, since $dt \neq$ invariant

$\hookrightarrow m\dot{\vec{x}}$ is not part of a 4-vector !

($m =$ mass = invariant particle property = constant !)

Correct relativistic generalization with *4-velocity*:

$$\begin{aligned} u^\mu &= \frac{dx^\mu}{d\tau}, & d\tau &= dt \sqrt{1-v^2}, & v &= |\dot{\vec{x}}| \\ & & &= \text{proper time of the particle} \\ & & &= \text{time in particle rest frame ("intrinsic clock")} \\ &= \left(\frac{dt}{d\tau}, \frac{d\vec{x}}{d\tau}\right) = \left(\frac{dt}{d\tau}, \dot{\vec{x}} \frac{dt}{d\tau}\right) = (\gamma, \gamma\dot{\vec{x}}), & \gamma &= 1/\sqrt{1-v^2} \end{aligned} \quad (2.58)$$

Relativistic 4-momentum:

$$p^\mu = mu^\mu = (m\gamma, m\gamma\dot{\vec{x}}) = \text{4-vector} \quad (2.59)$$

\hookrightarrow invariant square: $p^2 = m^2\gamma^2 - m^2\gamma^2v^2 = m^2$

$$\Rightarrow p^0 = p_0 = \sqrt{m^2 + \vec{p}^2} = m \left(1 + \frac{\vec{p}^2}{2m^2} + \dots\right) \quad \text{for } |\vec{p}| \ll m$$

= relativistic (total) energy of particle with mass m

$\hookrightarrow E_0 = mc^2 =$ rest energy (restoring $c \neq 1$ here)

$$T = p_0 - m = \text{kinetic energy} \quad (2.60)$$

Comments:

- $p^\mu = mu^\mu$ can be directly derived from ansatz $\vec{p} = m\dot{\vec{x}} \cdot f(v)$,
demanding momentum conservation in collisions and $f(0) = 1$
 - \vec{p} follows from invariance of action $S = \int dt L$ for point particle
 - p^0 is Hamilton function of free particle = conserved
 - 4-momentum conservation directly follows from translational invariance of Lagrange function / action
- } see exercises !

Comment:

$c = 1, \hbar = 1 \Rightarrow$ All kinematical quantities are measured in the same unit:

$$[E] = [m] = [p] = [x^{-1}] = [t^{-1}]. \quad (2.61)$$

Useful units in high-energy elementary particle physics:

- energy unit:

$$[E] = \text{Giga electron Volt (GeV)}, \quad \text{GeV}/c^2 = 1.8 \times 10^{-24} \text{g}. \quad (2.62)$$

- unit of length:

$$[x] = \text{Fermi} = \text{fm}. \quad (2.63)$$

Relation between units: $\hbar c = 0.197 \text{GeV fm}$

Example: particle decay $\pi^- \rightarrow \mu^- \bar{\nu}_\mu$
 $\uparrow \quad \quad \uparrow \quad \uparrow$
masses: $m_\pi \quad m_\mu \approx 0$

Rest frame of π^- :

$$p_\pi^\alpha = (m_\pi, \vec{0}), \quad p_\pi^2 = m_\pi^2, \quad (2.64)$$

$$p_\mu^\alpha = (E_\mu, \vec{p}_\mu), \quad p_\mu^2 = E_\mu^2 - \vec{p}_\mu^2 = m_\mu^2, \quad (2.65)$$

$$p_\nu^\alpha = (E_\nu, \vec{p}_\nu), \quad p_\nu^2 = E_\nu^2 - \vec{p}_\nu^2 = 0 \quad (2.66)$$

$$\text{energy conservation:} \quad m_\pi = E_\mu + E_\nu \quad (2.67)$$

$$\text{momentum conservation:} \quad \vec{0} = \vec{p}_\mu + \vec{p}_\nu \quad (2.68)$$

$$(2.68) \text{ in } (2.65) - (2.66): \quad m_\mu^2 = E_\mu^2 - E_\nu^2 = (E_\mu - E_\nu)(E_\mu + E_\nu) \quad (2.69)$$

$$\stackrel{(2.67)}{=} (E_\mu - E_\nu)m_\pi \quad (2.70)$$

$$\Rightarrow E_\mu = \frac{m_\pi}{2} + \frac{m_\mu^2}{2m_\pi}, \quad E_\nu = \frac{m_\pi}{2} - \frac{m_\mu^2}{2m_\pi} \quad (2.71)$$

Note: The direction of the decay is not fixed. π^- (= spin 0, no polarization!) decay isotropically in their rest frame.

Chapter 3

The Klein–Gordon equation

3.1 Relativistic wave equation

Non-relativistic quantum mechanics:

- Spinless particle (*scalar*)
↔ state vector $|\psi(t)\rangle \in$ Hilbert space,
 $\psi(t, \vec{x}) = \langle \vec{x} | \psi(t) \rangle =$ (complex) wave function in position representation
- Observables → Hermitian operators
Examples: position \hat{x} and momentum \hat{p} in position space
 $\hat{x} = \vec{x} =$ multiplicative, $\hat{p} = \frac{\hbar}{i} \vec{\nabla}$
- Correspondence principle: $H_{\text{classical}}(x_i, p_i) \rightarrow \hat{H}(\hat{x}_i, \hat{p}_i)$ with $[\hat{x}_i, \hat{p}_j] = i\hbar\delta_{ij}$
- Time evolution by Schrödinger equation: $i\hbar \frac{\partial}{\partial t} |\psi\rangle = \hat{H} |\psi\rangle$
- Free, spinless particle: $\hat{H} = \frac{\hat{p}^2}{2m}$
↔ Schrödinger equation is wave equation in position space:

$$i\hbar \frac{\partial}{\partial t} \psi(t, \vec{x}) = -\hbar^2 \frac{\vec{\nabla}^2}{2m} \psi(t, \vec{x}) \quad (3.1)$$

Note: wave equation invariant under Galilei trafo $\vec{x}' = R\vec{x} + \vec{a}$, $t' = t + \Delta t$
⇒ $\psi'(t, \vec{x}) = \psi(t - \Delta t, R^{-1}(\vec{x} - \vec{a}))$ is solution if $\psi(t, \vec{x})$ is.

Relativistic generalization:

Idea: $E \rightarrow i\hbar \frac{\partial}{\partial t}$ and $\vec{p} \rightarrow \frac{\hbar}{i} \vec{\nabla}$ in energy-momentum relation

- $E = \frac{\vec{p}^2}{2m} \Rightarrow$ Schrödinger equation
- $E = c\sqrt{\vec{p}^2 + (mc)^2} \Rightarrow$ problem with arbitrarily high spatial derivatives, no covariance !
- $E^2 = c^2\vec{p}^2 + (mc^2)^2 \Rightarrow -\frac{1}{c^2} \frac{\partial^2}{\partial t^2} \phi = \left(-\hat{\nabla}^2 + \underbrace{\left(\frac{mc}{\hbar}\right)^2}_{1 / (\text{reduced Compton wave length})} \right) \phi$

$$\xrightarrow{\hbar, c \rightarrow 1} (\partial_\mu \partial^\mu + m^2) \phi = (\square + m^2) \phi = 0 \quad \text{Klein–Gordon equation} \quad (3.2)$$

\hookrightarrow ansatz as wave equation for complex wave function ϕ

Relativistic covariance:

ϕ in two different frames of reference ($x' = \Lambda x + a$): $\phi'(x') = \phi'(\Lambda x + a) = \phi(x)$

- $0 = \underbrace{(\square + m^2)}_{\text{invariant}} \phi(x) = (\square' + m^2) \phi'(x') \Rightarrow$ form invariance of KG eq.
- $0 = (\square + m^2) \phi(x) = (\square + m^2) \phi'(\Lambda x + a)$
 $\Rightarrow \phi'(x) = \phi(\Lambda^{-1}(x - a))$ obeys KG eq. as well.

3.2 Solutions of the Klein–Gordon equation

Fourier ansatz in momentum space: (KG eq. = linear!)

$$\phi(x) = \phi(t, \vec{x}) = \int \frac{d^4 p}{(2\pi)^4} e^{-ipx} \tilde{\phi}(p)$$

$$\hookrightarrow (\square + m^2) \phi(x) = \int \frac{d^4 p}{(2\pi)^4} e^{-ipx} (-p^2 + m^2) \tilde{\phi}(p) = 0$$

\Rightarrow All $\tilde{\phi}(p)$ with $p^2 = m^2$ are solutions with $p^0 = \pm\omega_p$, where $\omega_p = +\sqrt{\vec{p}^2 + m^2}$.

Comment:
The sign ambiguity is due to the square of E in energy-momentum relation (genuine relat. feature!).

General solution:

$$\begin{aligned}
\phi(t, \vec{x}) &= \int \frac{d^4p}{(2\pi)^4} (2\pi) \underbrace{\delta(p^2 - m^2)}_{\text{ensures } p^2 = m^2} \left[\theta(+p_0) e^{-ipx} \underbrace{a(\vec{p})}_{\text{arbitrary complex functions}} + \theta(-p_0) e^{-ipx} \underbrace{b^*(-\vec{p})}_{\text{arbitrary complex functions}} \right] \\
&= \underbrace{\int \frac{d^4p}{(2\pi)^4} (2\pi) \delta(p^2 - m^2) \theta(+p_0)}_{\text{invariant phase-space volume}} \left[e^{-ipx} a(\vec{p}) + e^{+ipx} b^*(\vec{p}) \right] \\
&= \int \frac{d^3p}{2\omega_p (2\pi)^3} \equiv \int d\tilde{p} = \text{invariant phase-space volume} \\
&= \int d\tilde{p} \left[e^{-ipx} a(\vec{p}) + e^{+ipx} b^*(\vec{p}) \right] \tag{3.3}
\end{aligned}$$

Comment:
The integral $\int d^4p$ is Lorentz invariant, since $\int d^4p' = \int d^4p |\det(\Lambda)|$ with $|\det(\Lambda)| = 1$.
Moreover, $\text{sgn}(p'_0) = \text{sgn}(p_0)$ for $p' = \Lambda p$ and $p^2 \geq 0$.
Some explicit formulas:

$$\int \frac{d^4p}{(2\pi)^4} = \int \frac{d^3p}{(2\pi)^3} \int \frac{dp^0}{2\pi},$$

$$\int dp^0 \delta(p^2 - m^2) = \int dp^0 \delta((p^0)^2 - \vec{p}^2 - m^2) = \int dp^0 \frac{1}{2\omega_p} \left[\delta(p^0 + \omega_p) + \delta(p^0 - \omega_p) \right]$$

Note: Negative-energy solutions b^* raise problems.

- Energy spectrum $p^0 \in (-\infty, -m] \cup [m, \infty)$ not bounded from below.
 \hookrightarrow Particle can emit an infinite amount of energy (by perturbations).
 \Rightarrow System is unstable (no ground state) !
- Conversely, redefining $p^0 \geq 0$ leads to solutions with “wrong” time-evolution phase factor $e^{+ip^0 t}$ from non-relat. QM point of view.
- Setting $b(\vec{p}) \equiv 0$ is not consistent in presence of interactions.
 \hookrightarrow No solution of stability problem.
- QFT solves problem upon interpreting b^* solutions as antiparticles.
 $\hookrightarrow a, a^* (b^*, b)$ become annihilation/creation operators for (anti)particles.

3.3 Conserved current

Interpretation of ϕ as quantum mechanical wave function?

↔ Requirement: conserved “probability current” \vec{j} with probability density ρ , obeying $\dot{\rho} + \vec{\nabla} \cdot \vec{j} = 0$, so that $\int d^3x \rho(t, \vec{x}) = \text{const.}$

Recall non-relat. QM:

\vec{j}, ρ derived from Schrödinger equation and its conjugate:

$$\begin{aligned}
 i \frac{\partial}{\partial t} \psi &= -\frac{1}{2m} \vec{\nabla}^2 \psi, & -i \frac{\partial}{\partial t} \psi^* &= -\frac{1}{2m} \vec{\nabla}^2 \psi^*, \\
 \Rightarrow i \left[\psi^* \frac{\partial}{\partial t} \psi + \psi \frac{\partial}{\partial t} \psi^* \right] &= -\frac{1}{2m} \left[\psi^* \vec{\nabla}^2 \psi - \psi \vec{\nabla}^2 \psi^* \right] \\
 &\Rightarrow \frac{\partial}{\partial t} \underbrace{|\psi|^2}_{=\rho} = \underbrace{\vec{\nabla} \cdot \left[\frac{i}{2m} (\psi^* \vec{\nabla} \psi - \psi \vec{\nabla} \psi^*) \right]}_{=-\vec{j}}.
 \end{aligned} \tag{3.4}$$

Analogous manipulation with KG equation:

$$\begin{aligned}
 (\partial_\mu \partial^\mu + m^2) \phi &= 0, & (\partial_\mu \partial^\mu + m^2) \phi^* &= 0, \\
 \Rightarrow 0 &= \phi^* (\partial_\mu \partial^\mu + m^2) \phi - \phi (\partial_\mu \partial^\mu + m^2) \phi^* \\
 &= \partial_\mu \underbrace{[\phi^* \partial^\mu \phi - \phi \partial^\mu \phi^*]}_{=-2mi j^\mu}, & \text{continuity equation } \checkmark
 \end{aligned} \tag{3.5}$$

⇒ Conserved current: $j^\mu = \frac{i}{2m} [\phi^* \partial^\mu \phi - \phi \partial^\mu \phi^*]$, with $\partial^\mu j_\mu = 0$

$\rho \stackrel{?}{=} j^0 = \frac{i}{2m} [\phi^* \partial^0 \phi - \phi \partial^0 \phi^*] = \text{acceptable probability density ?}$

↔ Problem: ρ can become negative, i.e. $\rho \neq \text{probability density}$

QFT solution:

Conserved current $\propto j^\mu = (\rho, \vec{j})$ interpreted as charge (ρ) and current (\vec{j}) density of electric (or generalized) charge.

3.4 Interpretation of the Klein–Gordon equation

Clashes between principles of non-relat. QM and KG eq. / relat. covariance:

- Special relativity: space and time should be treated on equal footing.
QM: position is treated as an operator, time as a parameter.
- Negative energy solutions of the KG eq.
- Conserved density j^0 cannot be interpreted as probability density.
- Fields ϕ_μ transforming as 4-vectors under Lorentz transformation cannot be interpreted as wave functions as the matrices Λ for boosts are not unitary.
- Non-vanishing probability for propagation over space-like distances:

Assume resolution of momentum \vec{p} : $\Delta p^i \leq mc$

\hookrightarrow localization of a particle only within $\Delta x^i \sim \frac{\hbar}{\Delta p^i} \gtrsim \frac{\hbar c}{mc^2}$

\hookrightarrow probability $\neq 0$ for propagation between space-like separated events a and b
if $(x_a - x_b)^2 \sim (\hbar/mc)^2$

\Rightarrow Violation of causality due to quantum fluctuations ?

Outlook to solutions by relat. QFT:

- Field $\phi(t, \vec{x})$ satisfies the (covariant!) KG equation.
- \vec{x} and t are both treated as parameters.
 \hookrightarrow Elimination of the asymmetry of space and time.
- For a quantum mechanical description, the field is promoted to an operator:

$$\phi(t, \vec{x}) \rightarrow \hat{\phi}(t, \vec{x})$$

acting on particle states \in Hilbert space:

- action of $\hat{\phi}^\dagger(x)$ creates a particle / annihilates an antiparticle at point x ;
- action of $\hat{\phi}(x)$ creates an antiparticle / annihilates a particle at point x .

\Rightarrow QFT naturally becomes a many-particle theory.

\hookrightarrow Formalization best done within Lagrangian approach to continuum mechanics

Chapter 4

Classical Field Theory

4.1 Lagrangian and Hamiltonian formalism

4.1.1 Lagrangian field theory

Recapitulation of classical mechanics of point particle

- generalized coordinates $q_i = q_i(t)$ and velocities $\dot{q}_i = \frac{dq_i}{dt}$
- action $S = \int_{t_a}^{t_b} dt \underbrace{L(q_i, \dot{q}_i, t)}_{\text{Lagrange function}}$
- Hamilton's principle: $S = \text{extremal}$, i.e. $\delta S = 0$, with respect to the variations

$$q_i(t) \rightarrow q_i(t) + \delta q_i(t), \quad \dot{q}_i(t) \rightarrow \dot{q}_i(t) + \delta \dot{q}_i(t), \quad \delta \dot{q}_i(t) \equiv \frac{d}{dt} \delta q_i(t),$$

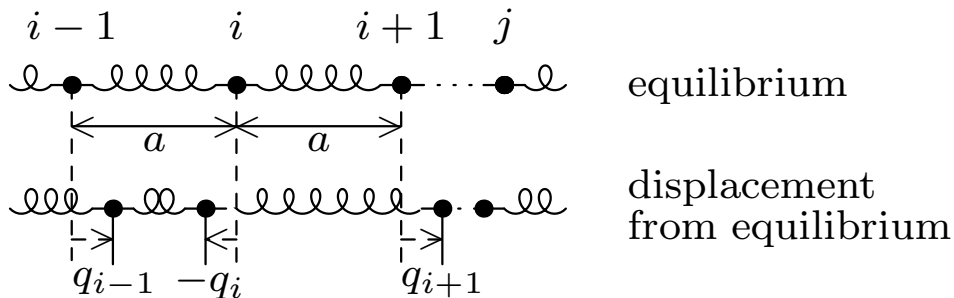
with the boundary conditions: $\delta q_i(t_{a/b}) = 0$

$$\Rightarrow \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0, \quad \text{Euler-Lagrange equations of motion}$$

From discrete to continuous systems:

Example (see e.g. Ref. [9]):

chain of equal mass points with mass m connected with massless uniform springs with force constant D (coupled harmonic oscillators for small q_i):



- kinetic energy: $T = \frac{1}{2} \sum_i m \dot{q}_i^2$
- potential energy: $V = \frac{1}{2} \sum_i D (q_{i+1} - q_i)^2$, $|q_{i+1} - q_i| = \text{extension length}$
- Lagrangian: $L = T - V = \sum_i a \underbrace{\left[\frac{m}{2a} \dot{q}_i^2 - \frac{a}{2} D \left(\frac{q_{i+1} - q_i}{a} \right)^2 \right]}_{= L_i}$
- Euler–Lagrange equations for coordinate q_i :

$$\underbrace{\frac{m}{a}}_{= \mu} \ddot{q}_i - \underbrace{Da}_{= Y} \underbrace{\frac{q_{i+1} - q_i}{a}}_{= s_i} + Da \frac{q_i - q_{i-1}}{a} = 0$$

= mass / length = extension / length

Note:

The force / length on an elastic rod is $f = Ys$ with $Y = \text{Young modulus (constant!)}$.

- Continuum limit:

$$\begin{aligned} \text{discrete } i &\rightarrow \text{continuous } x, & \sum_i a &\rightarrow \int dx \\ \frac{q_{i+1}(t) - q_i(t)}{a} &= \frac{q(t, x+a) - q(t, x)}{a} & \xrightarrow{a \rightarrow dx} & \frac{\partial q}{\partial x} \\ \Rightarrow L &= \int dx \frac{1}{2} \underbrace{\left[\mu \dot{q}(t, x)^2 - Y \left(\frac{\partial q}{\partial x} \right)^2 \right]}_{= \mathcal{L} = \text{Lagrangian density}} \end{aligned}$$

Comments:

- \mathcal{L} does not only depend on $q(x)$ and $\dot{q}(x)$, but also on $\frac{\partial q}{\partial x}$ due to nearest neighbour interaction (2nd spatial derivative \rightarrow next-to-nearest neighbour interaction, etc.).
- In more dimensions $q(t, x)$ is generalized to the field $\phi(t, \vec{x})$.

Generalization to fields:

- generalized coordinates:

$$\begin{array}{ccc} q_i(t) & \xrightarrow{\text{continuum limit}} & \phi(t, \vec{x}) = \phi(x) = \text{dyn. degree of freedom at } x^\mu = (t, \vec{x}) \\ \text{discrete} & & \text{continuous} \\ \text{index } i & & \text{"label" } \vec{x} \end{array}$$

Note: ϕ may carry more indices (for spin or other d.o.f.).

- Lagrange function:

$$L[\phi, \dot{\phi}] = \int d^3x \underbrace{\mathcal{L}(\phi, \dot{\phi}, \vec{\nabla}\phi, \dots)}_{\text{Lagrangian density, "Lagrangian"}} = \text{functional of } \phi \text{ and } \dot{\phi},$$

i.e. L maps $\phi, \dot{\phi}$ to a number (dependent on t , but not on \vec{x})

- action S in relativistic theories:

$$- S = S[\phi] = \int dt L[\phi, \dot{\phi}] = \text{functional of field } \phi, \text{ i.e. } S \text{ maps } \phi(x) \text{ to a constant.}$$

$$- S \stackrel{!}{=} \text{Lorentz invariant (= scalar)}$$

$$= \int dt L[\phi, \dot{\phi}] = \underbrace{\int d^4x \mathcal{L}(\phi, \dot{\phi}, \vec{\nabla}\phi, \dots)}_{\text{Lorentz invariant}}$$

$$\Rightarrow \mathcal{L} = \mathcal{L}(\phi, \dot{\phi}, \vec{\nabla}\phi, \dots) = \mathcal{L}(\phi, \partial\phi, \dots) = \text{Lorentz scalar}$$

$$- \int d^4x \text{ extends over complete Minkowski space with } |\phi| \rightarrow 0 \text{ sufficiently fast for } |x^\mu| \rightarrow \infty.$$

$$- S[\phi] \text{ is invariant under the transformation } \mathcal{L} \rightarrow \mathcal{L} + \partial^\mu F_\mu(\phi, \partial\phi, \dots), \text{ since surface terms vanish.}$$

$$\Leftrightarrow \mathcal{L} \text{ is unique up to partial integration.}$$

- Hamilton's principle:

$\delta S = 0$ under variation $\phi(x) \rightarrow \phi(x) + \delta\phi(x)$ with arbitrary infinitesimal $\delta\phi(x)$ vanishing at infinity:

$$0 = \delta S = \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi(x)} \delta\phi(x) + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi(x))} \partial_\mu \delta\phi(x) \right]$$

$$\stackrel{\text{part. int.}}{=} \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi(x)} - \partial_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi(x))} \right] \delta\phi(x), \quad (4.1)$$

$$\Rightarrow \frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} = 0 \quad \text{Euler-Lagrange equations for fields} \quad (4.2)$$

- Generalization to higher derivatives: recall *variational derivative*

$$\frac{\delta \mathcal{L}}{\delta \phi} = \frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} + \underbrace{\partial_\mu \partial_\nu \frac{\partial \mathcal{L}}{\partial (\partial_\mu \partial_\nu \phi)} + \dots}_{\text{only relevant for higher-order derivatives}} \quad (4.3)$$

$$\Rightarrow \text{Equation of motion (EOM): } \frac{\delta \mathcal{L}}{\delta \phi} = 0$$

Comment:

Derivation of the EOM via *functional derivative*:

$$\frac{\delta F[\phi]}{\delta \phi(x)} = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} (F[\phi(y) + \epsilon \delta(x-y)] - F[\phi(y)]) \quad (4.4)$$

Application to action functional:

$$S[\phi] = \int d^4y \mathcal{L}(\phi(y), \partial\phi(y)) \quad (4.5)$$

$$\begin{aligned} \Rightarrow \frac{\delta S[\phi]}{\delta \phi(x)} &= \int d^4y \left\{ \frac{\partial \mathcal{L}}{\partial \phi(y)} \delta(x-y) + \frac{\partial \mathcal{L}}{\partial (\partial_\mu^y \phi(y))} \partial_\mu^y \delta(x-y) \right\} \\ &= \frac{\partial \mathcal{L}}{\partial \phi(x)} - \partial_\mu^x \frac{\partial \mathcal{L}}{\partial (\partial_\mu^x \phi(x))} \end{aligned} \quad (4.6)$$

$$\Rightarrow \text{EOM: } \frac{\delta S[\phi]}{\delta \phi(x)} = 0 \quad (4.7)$$

Rules for functional derivatives:

$F, G = \text{functionals}; \quad a, b, c = \text{functions}$

- $\frac{\delta}{\delta \phi(x)} \phi(y) = \delta(x-y), \quad \frac{\delta}{\delta \phi(x)} a(y) = 0$
- $\frac{\delta}{\delta \phi(x)} (aF[\phi] + bG[\phi]) = a \frac{\delta F[\phi]}{\delta \phi(x)} + b \frac{\delta G[\phi]}{\delta \phi(x)},$
- $\frac{\delta}{\delta \phi(x)} (F[\phi]G[\phi]) = \frac{\delta F[\phi]}{\delta \phi(x)} G[\phi] + F[\phi] \frac{\delta G[\phi]}{\delta \phi(x)}.$

Relation between variational and functional derivative

\hookrightarrow consider function $f(\phi(x))$ of field $\phi(x)$ as specific type of functional

$$\frac{\delta}{\delta \phi(x)} f(\phi(y)) = \underbrace{\frac{\delta f}{\delta \phi}(\phi(x))}_{\text{variational derivative}} \delta(x-y) \equiv \frac{\delta f}{\delta \phi} \delta(x-y) \quad (4.8)$$

as function of $\phi(x)$

4.1.2 Hamiltonian field theory

- canonically conjugated momenta:

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \quad \longrightarrow \quad \pi(x) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}(x)} \quad (4.9)$$

point particles fields

- Hamilton function and density:

$$H(q_i, p_i) = \sum_i p_i \dot{q}_i - L \quad \longrightarrow \quad H = H(t) = \int d^3x \underbrace{\mathcal{H}(\phi, \vec{\nabla}\phi, \pi)}_{\substack{\text{Hamilton density,} \\ \text{Hamiltonian}}} \quad (4.10)$$

$$\text{with } \mathcal{H}(\phi, \vec{\nabla}\phi, \pi) = \pi \dot{\phi} - \mathcal{L} \quad (4.11)$$

- Hamiltonian EOMs:

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} \quad \longrightarrow \quad \dot{\phi} = \frac{\delta \mathcal{H}}{\delta \pi}, \quad \dot{\pi} = -\frac{\delta \mathcal{H}}{\delta \phi} \quad (4.12)$$

Note: \mathcal{H} and $\dot{\phi}$ depend on ϕ , $\vec{\nabla}\phi$, and π (but not on derivatives of π).

Derivation of Hamiltonian EOMs:

$$\frac{\delta \mathcal{H}}{\delta \pi} = \frac{\partial \mathcal{H}}{\partial \pi} = \dot{\phi} + \pi \underbrace{\frac{\partial \dot{\phi}}{\partial \pi}}_{=\pi} - \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \frac{\partial \dot{\phi}}{\partial \pi} = \dot{\phi}, \quad (4.13)$$

$$\begin{aligned} \frac{\delta \mathcal{H}}{\delta \phi} &= \frac{\partial \mathcal{H}}{\partial \phi} - \vec{\nabla} \frac{\partial \mathcal{H}}{\partial \vec{\nabla}\phi} = \pi \frac{\partial \dot{\phi}}{\partial \phi} - \frac{\partial \mathcal{L}}{\partial \phi} - \underbrace{\frac{\partial \mathcal{L}}{\partial \dot{\phi}} \frac{\partial \dot{\phi}}{\partial \phi}}_{=\pi} - \vec{\nabla} \frac{\partial \mathcal{H}}{\partial \vec{\nabla}\phi} \\ &= -\frac{\partial \mathcal{L}}{\partial \phi} - \vec{\nabla} \frac{\partial \mathcal{H}}{\partial \vec{\nabla}\phi} \stackrel{\text{Lag. EOM}}{=} -\partial_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} - \vec{\nabla} \frac{\partial \mathcal{H}}{\partial \vec{\nabla}\phi} \\ &= -\frac{\partial}{\partial t} \underbrace{\frac{\partial \mathcal{L}}{\partial \dot{\phi}}}_{=\pi} - \vec{\nabla} \frac{\partial \mathcal{L}}{\partial \vec{\nabla}\phi} - \vec{\nabla} \frac{\partial \mathcal{H}}{\partial \vec{\nabla}\phi} = -\dot{\pi}, \end{aligned}$$

$$\text{because } \left. \frac{\partial \mathcal{H}}{\partial \vec{\nabla}\phi} \right|_{\pi, \phi \text{ fixed}} = \pi \frac{\partial \dot{\phi}}{\partial \vec{\nabla}\phi} - \frac{\partial \mathcal{L}}{\partial \vec{\nabla}\phi} - \underbrace{\frac{\partial \mathcal{L}}{\partial \dot{\phi}} \frac{\partial \dot{\phi}}{\partial \vec{\nabla}\phi}}_{=\pi} = -\left. \frac{\partial \mathcal{L}}{\partial \vec{\nabla}\phi} \right|_{\phi, \phi \text{ fixed}}. \quad \text{q.e.d.}$$

4.2 Actions for scalar fields

Question: Which Lagrangian leads to the Klein–Gordon eq. $(\partial_\mu\partial^\mu + m^2)\phi = 0$, where $\phi(x) =$ real or complex scalar field ?

4.2.1 Free real scalar field

The Lagrangian

Requirements on \mathcal{L} :

- KG eq. is linear in ϕ ; EOMs reduce \mathcal{L} by one power in ϕ .
 $\hookrightarrow \mathcal{L}$ is bilinear in ϕ and its derivatives $\partial_\mu\phi$, etc.
- KG involves only derivatives up to 2nd order.
 $\hookrightarrow \mathcal{L}$ contains only derivatives up to 2nd order.
- $\mathcal{L} =$ Lorentz invariant.
 $\hookrightarrow \mathcal{L}$ is linear combination of bilinear, Lorentz-invariant terms formed by ϕ , $\partial_\mu\phi$, etc.
 \Rightarrow All possible terms: ϕ^2 , $\phi\Box\phi$, $(\partial_\mu\phi)(\partial^\mu\phi)$.
 But: $\phi\Box\phi = \underbrace{\partial_\mu(\phi\partial^\mu\phi)}_{\text{irrelevant surface term}} - (\partial_\mu\phi)(\partial^\mu\phi)$ can be omitted.
- $\mathcal{L} =$ real.

\Rightarrow Most general ansatz:

$$\mathcal{L} = A(\partial_\mu\phi)(\partial^\mu\phi) + B\phi^2 = Ag^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi) + B\phi^2 \quad \text{with } A, B = \text{real}$$

Determine A, B by EOM:

$$0 = \partial_\rho \frac{\partial \mathcal{L}}{\partial(\partial_\rho\phi)} - \frac{\partial \mathcal{L}}{\partial\phi} = \partial_\rho (Ag^{\mu\nu}\delta_\mu^\rho\partial_\nu\phi + Ag^{\mu\nu}\delta_\nu^\rho\partial_\mu\phi) - 2B\phi = 2A \left[\partial_\rho\partial^\rho\phi - \frac{B}{A}\phi \right] \quad (4.14)$$

$$\hookrightarrow B/A \stackrel{!}{=} -m^2, \quad A = 1/2 \quad (= \text{convention, see } H \text{ below})$$

\Rightarrow Lagrangian for free scalar field:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2, \quad \text{alternatively: } \mathcal{L} = -\frac{1}{2}\phi(\Box + m^2)\phi \quad (4.15)$$

Hamiltonian formalism:

- Canonically conjugated field: $\pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi}$

$$\hookrightarrow \text{Hamiltonian: } \mathcal{H} = \pi \dot{\phi} - \mathcal{L} = \frac{1}{2} \left[\pi^2 + (\vec{\nabla} \phi)^2 + m^2 \phi^2 \right]$$

- Explicit solution:

$$\begin{aligned} \phi(x) &= \int d\vec{p} \left[e^{-ipx} a(\vec{p}) + e^{+ipx} b^*(\vec{p}) \right] \Big|_{p_0 = +\sqrt{\vec{p}^2 + m^2}} \\ \pi(x) &= -i \int d\vec{p} p_0 \left[e^{-ipx} a(\vec{p}) - e^{+ipx} b^*(\vec{p}) \right] \Big|_{p_0 = +\sqrt{\vec{p}^2 + m^2}} \end{aligned} \quad (4.16)$$

Note: $b(\vec{p}) = a(\vec{p})$ for a real scalar field.

\hookrightarrow Hamiltonian:

$$\begin{aligned} H &= \int d^3x \mathcal{H} = \frac{1}{2} \int d^3x \int d\vec{p} \int d\vec{q} \left\{ [-p_0 q_0 - \vec{p} \cdot \vec{q} + m^2] (a(\vec{p}) a(\vec{q}) e^{-i(p+q) \cdot x} + \text{c.c.}) \right. \\ &\quad \left. + [p_0 q_0 + \vec{p} \cdot \vec{q} + m^2] (a(\vec{p}) a^*(\vec{q}) e^{-i(p-q) \cdot x} + \text{c.c.}) \right\} \\ &\quad \text{Use identity } \int d^3x e^{i\vec{k} \cdot \vec{x}} = (2\pi)^3 \delta(\vec{k}) \\ &= \frac{1}{2} \int \frac{d^3p}{(2\pi)^3 (2p_0)^2} \left\{ \underbrace{[-p_0^2 + \vec{p}^2 + m^2]}_{=0} (a(\vec{p}) a(-\vec{p}) e^{-2ip^0 x^0} + \text{c.c.}) \right. \\ &\quad \left. + [p_0^2 + \vec{p}^2 + m^2] (a(\vec{p}) a^*(\vec{p}) + \text{c.c.}) \right\} \Big|_{p_0 = \sqrt{\vec{p}^2 + m^2}} \\ &= \int d\vec{p} \sqrt{\vec{p}^2 + m^2} |a(\vec{p})|^2 = \text{const} > 0 \quad \checkmark \end{aligned} \quad (4.17)$$

\Rightarrow Hamiltonian fulfills requirement on kinetic energy (constant, non-negative).

4.2.2 Free complex scalar field

- Complex scalar field ϕ can be decomposed: $\phi = \frac{1}{\sqrt{2}} (\phi_1 + i\phi_2)$ with $\phi_1, \phi_2 = \text{real}$

$$(\square + m^2)\phi_i = 0, \quad i = 1, 2 \quad \Leftrightarrow \quad (\square + m^2)\phi = 0, \quad (\square + m^2)\phi^* = 0 \quad (4.18)$$

- Lagrangian (2 free real fields $\hat{=}$ 1 complex field):

$$\begin{aligned} \mathcal{L} &= \frac{1}{2} [(\partial_\mu \phi_1)(\partial^\mu \phi_1) - m^2 \phi_1 \phi_1] + \frac{1}{2} [(\partial_\mu \phi_2)(\partial^\mu \phi_2) - m^2 \phi_2 \phi_2] \\ &= (\partial_\mu \phi^*)(\partial^\mu \phi) - m^2 \phi^* \phi, \quad \text{alternatively: } \mathcal{L} = -\phi^*(\square + m^2)\phi \end{aligned} \quad (4.19)$$

- EOMs from variations of $\phi^{(*)}$:

$$0 = \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)} - \frac{\partial \mathcal{L}}{\partial \phi^*} = (\partial_\mu \partial^\mu + m^2) \phi, \quad (4.20)$$

$$0 = \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} - \frac{\partial \mathcal{L}}{\partial \phi} = (\partial_\mu \partial^\mu + m^2) \phi^* \quad (4.21)$$

- Conjugate fields and Hamiltonian:

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi}^*, \quad \pi^* = \frac{\partial \mathcal{L}}{\partial \dot{\phi}^*} = \dot{\phi} \quad (4.22)$$

$$\hookrightarrow \mathcal{H} = \pi \dot{\phi} + \pi^* \dot{\phi}^* - \mathcal{L} = |\pi|^2 + |\vec{\nabla} \phi|^2 + m^2 |\phi|^2 \quad (4.23)$$

- Explicit solutions as in (4.16), but with $a(\vec{p}) \neq b(\vec{p})$

\hookrightarrow Hamiltonian:

$$H = \dots = \int d\tilde{p} \sqrt{\vec{p}^2 + m^2} [a(\vec{p}) a^*(\vec{p}) + b^*(\vec{p}) b(\vec{p})] = \text{const} > 0 \quad \checkmark \quad (4.24)$$

4.3 Interacting fields

4.3.1 Scalar self-interactions

Extension analogous to $L = T - V$ in point mechanics: (real ϕ as example)

$$\mathcal{L} = \underbrace{\frac{1}{2} (\partial_\mu \phi) (\partial^\mu \phi)}_{\text{kinetic term}} - \underbrace{\frac{m^2}{2} \phi^2 - V(\phi)}_{\text{potential term}} \quad (4.25)$$

$$\text{with } V = \underbrace{c_0}_{=0} + \underbrace{c_1}_{=0} \phi + c_3 \phi^3 + c_4 \phi^4 + \dots \quad (4.26)$$

Comments:

- $c_0 = 0$: arbitrary definition of energy offset
- $c_1 = 0$: arbitrary offset in ϕ , such that V is minimal in ground state $\phi \equiv 0$
($V \rightarrow +\infty$ for $|\phi| \rightarrow \infty$, otherwise system unstable. $\rightarrow V$ has minimum.)
- $c_2 = \frac{1}{2} m^2 \geq 0$: otherwise no minimum of V at $\phi \equiv 0$
- Dim. analysis: $[A] = \text{unit of } A$, $\dim[A] = \text{mass dimension of } A$
 $\Rightarrow [\text{Action } S] = [\hbar] = 1$, $\dim[d^4 x] = -4$, $\dim[\partial] = +1 = \text{dim. of mass}$
 $\Rightarrow \dim[\mathcal{L}] = \dim[V] = 4$, $\dim[\phi] = 1$, $\dim[c_3] = 1$, $\dim[c_4] = 0$, $\dim[c_5] = -1, \dots$

- Convenient convention:

$$c_n = g_n/\Lambda^{4-n} \text{ with } g_n = \text{dimensionless and } \Lambda = \text{common mass scale}$$

- QFT: Theories with $\dim[\text{couplings}] < 0$ are *non-renormalizable*, i.e. some observables diverge due to short-distance interactions (UV limit).

But: Such theories can still be useful as low-energy *effective field theories*, where momenta $> \Lambda$ (distances $< 1/\Lambda$) are excluded.

- Non-renormalizable interactions can also involve derivatives of order > 2 .

EOM and its Green function:

$$\frac{\partial \mathcal{L}}{\partial \phi} = -m^2 \phi - \frac{\partial V}{\partial \phi}, \quad \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \partial^\mu \phi$$

\Rightarrow KG eq. with interaction:

$$0 = \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} - \frac{\partial \mathcal{L}}{\partial \phi} = \square \phi + m^2 \phi + \frac{\partial V}{\partial \phi}, \quad (4.27)$$

Non-linear 2nd order partial differential equation

\leftrightarrow define Green function $D(x, y)$:

$$(\square_x + m^2) D(x, y) = -\delta^4(x - y). \quad (4.28)$$

\leftrightarrow integral equation equivalent to (4.27):

$$\Rightarrow \phi(x) = \underbrace{\phi_0(x)}_{\text{solution of free KG eq.,}} + \int d^4 y D(x, y) \frac{\partial V(\phi(y))}{\partial \phi} \quad (4.29)$$

$$(\square + m^2)\phi_0 = 0$$

Check:

$$(\square_x + m^2)\phi(x) = \underbrace{(\square_x + m^2)\phi_0(x)}_{=0} + \int d^4 y \underbrace{(\square_x + m^2)D(x, y)}_{-\delta^4(x-y)} \frac{\partial V(\phi(y))}{\partial \phi} = -\frac{\partial V(\phi(x))}{\partial \phi}$$

Iterative solution for sufficiently weak interaction: (*perturbation theory*)

- 0th approximation: free motion

$$\phi(x) = \phi_0(x) \quad (4.30)$$

- 1st approximation: insert $\phi = \phi_0$ in r.h.s. of (4.29) \rightarrow *Born approximation*

$$\phi_1(x) = \phi_0(x) + \int d^4y D(x, y) \frac{\partial V(\phi_0(y))}{\partial \phi} \quad (4.31)$$

- 2nd approximation: inserting $\phi = \phi_1$ in r.h.s. of (4.29)

$$\begin{aligned} \phi_2(x) &= \phi_0(x) + \int d^4y D(x, y) \frac{\partial V(\phi_1(y))}{\partial \phi} \\ &= \phi_0(x) + \int d^4y D(x, y) \frac{\partial V(\phi_0(y))}{\partial \phi} \\ &\quad + \int d^4y_1 \frac{\partial^2 V(\phi_0(y_1))}{\partial \phi^2} D(x, y_1) \int d^4y_2 D(y_1, y_2) \frac{\partial V(\phi_0(y_2))}{\partial \phi} + \dots, \end{aligned} \quad (4.32)$$

(Expansion to 2nd order in potential terms \rightarrow singles out correction to Born approx.)

- n -th approximation: insert $\phi = \phi_{n-1}$ in r.h.s. of (4.29)

Visualization of n th correction:

free propagation between n local interactions with V at space-time points y_i .

\hookrightarrow Hope that

$$\phi_n(x) \xrightarrow{n \rightarrow \infty} \phi(x). \quad (4.33)$$

4.3.2 Explicit calculation of the Green function (*propagator*)

Defining Eq. (4.28) = linear, inhomogenous diff. eq.

\hookrightarrow Fourier ansatz:

$$D(x, y) = \int \frac{d^4k}{(2\pi)^4} D(k) e^{-ik \cdot (x-y)} \quad (4.34)$$

Note: $D(x, y) = D(x - y)$ because of translational invariance

Insertion of ansatz into (4.28):

$$\begin{aligned} (\square_x + m^2)D(x, y) &= \int \frac{d^4k}{(2\pi)^4} D(k) (-k^2 + m^2) e^{-ik \cdot (x-y)} \\ &\stackrel{!}{=} -\delta^4(x - y) = - \int \frac{d^4k}{(2\pi)^4} e^{-ik \cdot (x-y)} \end{aligned} \quad (4.35)$$

$$\begin{aligned} \Rightarrow D(k) &= \frac{1}{k^2 - m^2} = \frac{1}{\left(k^0 - \sqrt{\vec{k}^2 + m^2}\right) \left(k^0 + \sqrt{\vec{k}^2 + m^2}\right)} \\ &= \frac{1}{2\sqrt{\vec{k}^2 + m^2}} \left[\frac{1}{k^0 - k_0^+} - \frac{1}{k^0 - k_0^-} \right] \quad \text{with } k_0^\pm = \pm \sqrt{\vec{k}^2 + m^2} \end{aligned} \quad (4.36)$$

$$\Rightarrow D(x, y) = \int d\tilde{k} e^{i\tilde{k}(\tilde{x}-\tilde{y})} \int_{-\infty}^{\infty} \frac{dk^0}{2\pi} \left[\frac{1}{k^0 - k_0^+} - \frac{1}{k^0 - k_0^-} \right] e^{-ik^0(x^0 - y^0)} \quad (4.37)$$

Note: Prescriptions needed to resolve convergence problem near poles at $k^0 = k_0^\pm$!

Solution:

Move the poles into the complex plane by an infinitesimal shift $i\delta$ ($\delta > 0$) and use identity

$$\int_{-\infty}^{\infty} d\kappa \frac{e^{-i\kappa x}}{\kappa \pm i\delta} = \mp 2\pi i \theta(\pm x) \quad (4.38)$$

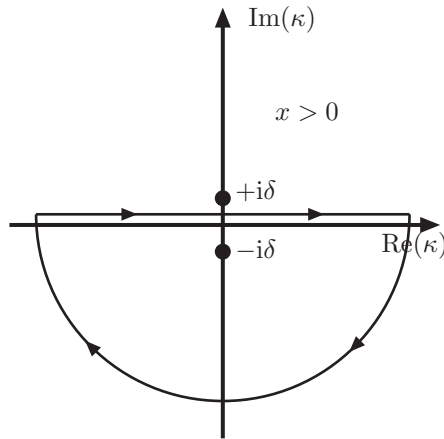
Comment:

Prove of identity with residue theorem:

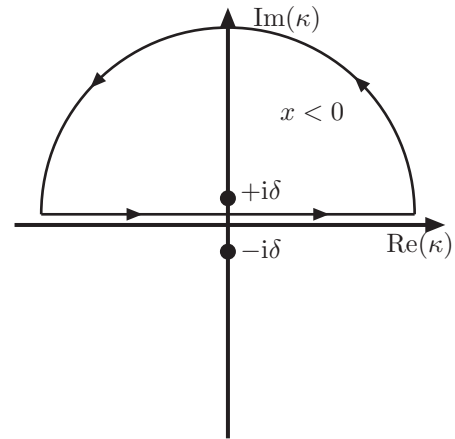
Interpret $\int d\kappa$ as line integral in complex κ -plane and close contour with half-circle of infinite radius in such a way that half-circle does not contribute:

Integrand on half-circle $\propto \exp\{\text{Im}(\kappa)x\} \Rightarrow$ damping for $\text{Im}(\kappa)x < 0$.

\Rightarrow Close contour in lower (upper) half-plane for $x > 0$ ($x < 0$).



$$\oint d\kappa \frac{e^{-i\kappa x}}{\kappa \pm i\delta} = \begin{cases} -2\pi i, \\ 0. \end{cases}$$



$$\oint d\kappa \frac{e^{-i\kappa x}}{\kappa \pm i\delta} = \begin{cases} 0, \\ +2\pi i. \end{cases}$$

q.e.d.

Application of Eq. (4.38):

$$\int_{-\infty}^{\infty} \frac{dk^0}{2\pi} \frac{e^{-ik^0(x^0 - y^0)}}{k^0 - k_0^\sigma \pm i\delta} = \mp i e^{-ik_0^\sigma(x^0 - y^0)} \theta(\pm(x^0 - y^0)) \quad \text{with } k_0^\sigma = k_0^+ \text{ or } k_0^- \quad (4.39)$$

\Rightarrow Poles at $k_0^\sigma - i\delta$ correspond to forward propagation in time (contribution only for $x^0 > y^0$);
poles at $k_0^\sigma + i\delta$ correspond to backward propagation in time (contribution only for $y^0 > x^0$).

\Rightarrow 4 different types of propagators:

- Poles at $k_0^\pm - i\delta$: *retarded propagator* \rightarrow forward propagation of all modes

$$D_{\text{ret}}(x, y) = -i\theta(x^0 - y^0) \int d\tilde{k} e^{-ik(x-y)} + \text{c.c.} = \text{real} \quad (4.40)$$

Properties: (non-trivial!)

- Causal behaviour: $D_{\text{ret}}(x, y) = 0$ for $(x - y)^2 < 0$
- Lorentz invariance

\hookrightarrow appropriate for causal wave propagation in classical field theory

- Poles at $k_0^\pm + i\delta$: *advanced propagator* \rightarrow backward propagation of all modes

$$D_{\text{adv}}(x, y) = D_{\text{ret}}(y, x) \quad (4.41)$$

\hookrightarrow backward propagation in classical field theory

- Poles at $k_0^+ - i\delta$ and $k_0^- + i\delta$: *Feynman propagator*
 $\hookrightarrow k_0^+$ with forward, k_0^- with backward propagation

$$\begin{aligned} D_{\text{F}}(x, y) &= \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik(x-y)}}{(k^0 - k_0^+ + i\delta)(k^0 - k_0^- - i\delta)}, \quad k_0^\pm \mp i\delta \equiv \pm\sqrt{\vec{k}^2 + m^2 - i\epsilon} \\ &= \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik(x-y)}}{k^2 - m^2 + i\epsilon} \quad (\text{infinitesimal } \epsilon > 0) \end{aligned} \quad (4.42)$$

$$= -i\theta(x^0 - y^0) \int d\tilde{k} e^{-ik \cdot (x-y)} - i\theta(y^0 - x^0) \int d\tilde{k} e^{+ik \cdot (x-y)} \quad (4.43)$$

$$= D_{\text{F}}(y, x) = \text{complex}$$

Properties:

- Lorentz invariance (obvious!), $\epsilon > 0$ acts like decay width for all modes
- Causal behaviour non-trivial:

$$D_{\text{F}}(x, y) \neq 0 \text{ for } s^2 = (x - y)^2 < 0 \quad (\text{exp. decay } \propto e^{-mr} \text{ with } r = \sqrt{-s^2}).$$

Causality restored by independence of qm. measurements at x, y with $s^2 < 0$.

- Propagator naturally appears in QFT:

$x^0 > y^0$: forward propagation of particles with $k^0 > 0$;

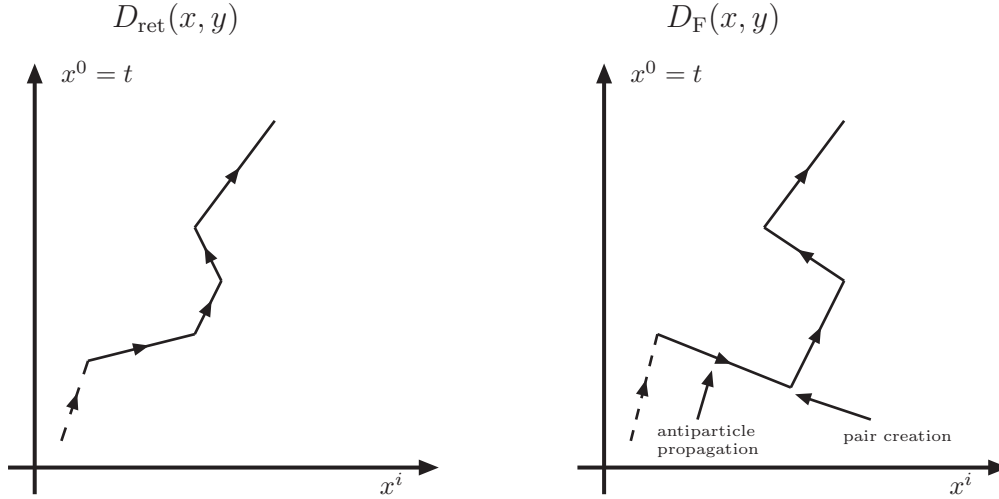
$y^0 > x^0$: backward propagation of particles with “ $k^0 < 0$ ”

\hookrightarrow reinterpreted as forward propagation of antiparticles

||| Comment:
 ||| No transport of information by acausal behaviour of $D_{\text{F}}(x, y)$.
 ||| Phenomenon similar to Einstein–Podolsky–Rosen paradox.

- Poles at $k_0^+ + i\delta$ and $k_0^- - i\delta$: Feynman propagator $D_F(x, y)^*$ for time-reversed QFT

Illustration of perturbative expansion for $\phi(x)$:



4.4 Symmetries and Noether Theorem

Noether theorem in classical mechanics:

Every continuous symmetry of a system leads to a conservation law, e.g.

- Rotational invariance \Rightarrow conservation of angular momentum.
- Translational invariance \Rightarrow cartesian momentum conservation.

Now: generalization to field theory.

4.4.1 Continuous symmetries

Definition:

A field theory possesses an infinitesimal continuous symmetry if a transformation

$$\phi_k \rightarrow \phi'_k \equiv \phi_k + \delta\omega_a \Delta_k^a(\phi), \tag{4.44}$$

leaves the action invariant,

$$S[\phi'] = S[\phi]. \tag{4.45}$$

Notation:

- $\delta\omega_a$ = infinitesimal parameters of the transformation
 = $\begin{cases} \text{const. for a } \textit{global} \text{ symmetry,} \\ \text{function}(x) \text{ for a } \textit{local} \text{ symmetry.} \end{cases}$

- $\Delta_k^a(\phi)$ = functions of all fields ϕ_k and their derivatives.
- Index a = “internal” index or Lorentz index (a may stand for multiple indices).
- Index k runs over all fields ϕ_k .

Implications for \mathcal{L} and S :

If all $\phi \rightarrow 0$ sufficiently fast for $|x^\mu| \rightarrow \infty$, the invariance (4.45) of S implies that \mathcal{L} can only change by a total derivative:

$$\delta\mathcal{L} \equiv \mathcal{L}(\phi') - \mathcal{L}(\phi) = \partial_\mu(K^{a,\mu}(\phi)\delta\omega_a) + \mathcal{O}(\delta\omega^2), \quad (4.46)$$

since

$$\begin{aligned} S[\phi'] &= S[\phi] + \underbrace{\int_V d^4x \partial_\mu(K^{a,\mu}(\phi)\delta\omega_a)}_{= \text{surface integral (Gauss!)} = 0} = S[\phi]. \end{aligned} \quad (4.47)$$

Examples for internal symmetries:

- $U(1)$ symmetry of a complex scalar theory:

$$\mathcal{L} = (\partial_\mu\phi^*)(\partial^\mu\phi) - m^2\phi^*\phi - V(\phi^*\phi) = \text{invariant under trafo}$$

$$\phi' = \exp\{-iq\omega\}\phi, \quad \text{i.e.} \quad \Delta(\phi) = -iq\phi \quad \text{with } q = \text{const.} \quad (4.48)$$

Note: If ϕ describes an electrically charged particle, $\phi \rightarrow \phi'$ is an elmng. gauge transformation with q being the electric charge.

- $SU(N)$ symmetry of N complex scalars: $\Phi = (\phi_1, \dots, \phi_N)^T$

$$\mathcal{L} = (\partial_\mu\Phi)^\dagger(\partial^\mu\Phi) - m^2\Phi^\dagger\Phi - V(\Phi^\dagger\Phi) = \text{invariant under trafo}$$

$$\Phi' = U\Phi, \quad \text{with } U^\dagger U = \mathbf{1}, \quad \underbrace{\det(U) = +1}_{U = \text{“special”}} \quad (4.49)$$

$$SU(N) = \text{group of all special, unitary } N \times N \text{ matrices } U.$$

Exponential parametrization of U and infinitesimal transformations:

$$\begin{aligned} U(\omega_a) &= \exp\{-igT^a\omega_a\}, \quad T^a = \text{generators} = \text{matrices}, \quad g = \text{const.} \\ U(\delta\omega_a) &= \mathbf{1} - igT^a\delta\omega_a + \dots, \quad \text{i.e.} \quad \Delta_k^a(\phi) = -igT_{kl}^a\phi_l \end{aligned} \quad (4.50)$$

Properties of T^a (since $U^\dagger = U^{-1}$):

$$\begin{aligned} U(\delta\omega_a)^{-1} &= U(-\delta\omega_a) = \mathbf{1} + igT^a\delta\omega_a + \dots \\ &= U(\delta\omega_a)^\dagger = \mathbf{1} + ig(T^a)^\dagger\delta\omega_a + \dots \quad \Rightarrow \quad T^a = (T^a)^\dagger = \text{hermitian}, \\ 1 &= \det(U) = \exp\{\text{Tr}(-igT^a\omega_a)\} \quad \Rightarrow \quad \text{Tr}(T^a) = T_{kk}^a = 0. \end{aligned} \quad (4.51)$$

$\Rightarrow a = 1, \dots, n = N^2 - 1 = \#$ independent traceless, hermitian $N \times N$ matrices.

- $SO(N)$ symmetry of N real scalars: $\Phi = (\phi_1, \dots, \phi_N)^T$

$\mathcal{L} = \frac{1}{2}(\partial_\mu \Phi)^T (\partial^\mu \Phi) - \frac{1}{2}m^2 \Phi^T \Phi - V(\Phi^T \Phi) =$ invariant under trafo

$$\begin{aligned} \Phi' &= R \Phi, \quad \text{with } R^T R = \mathbf{1}, \quad \det(R) = +1, \\ SO(N) &= \text{group of all special, orthogonal } N \times N \text{ matrices } R. \end{aligned} \quad (4.52)$$

Exponential parametrization of R and infinitesimal transformations:

$$R(\omega_a) = \exp\{-igT^a \omega_a\}, \quad \dots \quad \Delta_k^a(\phi) = -igT_{kl}^a \phi_l. \quad (4.53)$$

Properties of T^a (since $R^T = R^{-1}$):

$$T_{kl}^a = -T_{lk}^a = \text{imaginary}, \quad T_{kk}^a = 0, \quad a = 1, \dots, n = N(N-1)/2. \quad (4.54)$$

Space–time symmetries:

- Space–time translations $x^\mu \rightarrow x'^\mu = x^\mu + \omega^\mu$ with $\omega^\mu = \text{const.}$:

$$\phi(x) \rightarrow \phi'(x) = \phi(x - \omega) = \phi(x) - \omega_\mu \partial^\mu \phi(x) + \mathcal{O}(\omega^2) \quad \Rightarrow \quad \Delta^\mu(\phi) = -\partial^\mu \phi, \quad (4.55)$$

i.e. index a acts as Lorentz index μ .

Transformation of the Lagrangian:

$$\delta \mathcal{L} = \mathcal{L}(x - \omega) - \mathcal{L}(x) = -\omega_\nu \partial^\nu \mathcal{L} \equiv \omega_\nu \partial_\mu K^{\mu\nu} \quad \Rightarrow \quad K^{\mu\nu} = -g^{\mu\nu} \mathcal{L}. \quad (4.56)$$

4.4.2 Derivation of the Noether theorem

Noether theorem:

For each global symmetry of the action,

$$\phi_k \rightarrow \phi_k + \delta\omega_a \Delta_k^a(\phi), \quad \delta \mathcal{L} = \delta\omega_a \partial_\mu K^{a,\mu}, \quad \delta\omega_a = \text{const.}, \quad (4.57)$$

there is a set of conserved *Noether currents* j_a^μ ,

$$0 = \partial_\mu j_a^\mu = \partial_0 j_a^0 + \vec{\nabla} \cdot \vec{j}_a, \quad (4.58)$$

if the fields ϕ_k satisfy the EOMs.

Proof:

$$\begin{aligned} 0 &= \delta \mathcal{L} - \delta\omega_a \partial_\mu K^{a,\mu}(\phi) \\ &= \frac{\partial \mathcal{L}}{\partial \phi_k} \delta\omega_a \Delta_k^a(\phi) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_k)} \delta\omega_a \partial_\mu \Delta_k^a(\phi) - \delta\omega_a \partial_\mu K^{a,\mu}(\phi) \\ &\stackrel{\text{EOM}}{=} \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_k)} \delta\omega_a \Delta_k^a(\phi) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_k)} \delta\omega_a \partial_\mu \Delta_k^a(\phi) - \delta\omega_a \partial_\mu K^{a,\mu}(\phi) \right] \\ &= \delta\omega_a \partial_\mu \underbrace{\left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_k)} \Delta_k^a(\phi) - K^{a,\mu}(\phi) \right]}_{\equiv j^{a,\mu}} \end{aligned} \quad (4.59)$$

Note: A sum over repeated labels k is implied.

q.e.d.

Implications:

- Noether currents:

$$j^{a,\mu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_k)} \Delta_k^a(\phi) - K^{a,\mu}(\phi), \quad \partial j^a = 0. \quad (4.60)$$

Note: j^a only fixed up to a constant factor.

- Noether charges:

$$Q^a(t) \equiv \int d^3x j^{a,0}(t, \vec{x}) = \int d^3x \left[\underbrace{\pi_k}_{=\frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)}} \Delta_k^a(\phi) - K^{a,0}(\phi) \right]. \quad (4.61)$$

Charge conservation:

$$\dot{Q}^a(t) = \int_{V=\text{const.}} d^3x \partial_0 j^{a,0}(t, \vec{x}) = - \int_V d^3x \vec{\nabla} \cdot \vec{j}^a \stackrel{\text{Gauss}}{=} - \oint_{A(V)} d^2\vec{A} \cdot \vec{j}^a = 0, \quad (4.62)$$

if the currents j^a vanish sufficiently fast for $|\vec{x}| \rightarrow \infty$.

4.4.3 Internal symmetries and conserved currents

↔ Reconsider examples from Sect. 4.4.1

- $U(1)$ symmetry of the complex scalar theory:

$$\mathcal{L} = (\partial_\mu \phi^*)(\partial^\mu \phi) - m^2 \phi^* \phi - V(\phi^* \phi), \quad \phi' = \exp\{-iq\omega\}\phi, \quad \Delta(\phi) = -iq\phi. \quad (4.63)$$

Noether current:

$$\begin{aligned} j^\mu &= -iq \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \phi - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)} \phi^* \right) = -iq [(\partial^\mu \phi^*)\phi - \phi^*(\partial^\mu \phi)] \\ &= iq\phi^* \overleftrightarrow{\partial}^\mu \phi \quad \text{with} \quad f(x) \overleftrightarrow{\partial}_\mu g(x) \equiv f(x) \partial_\mu g(x) - (\partial_\mu f(x))g(x). \end{aligned} \quad (4.64)$$

Note: Result agrees (up to prefactor) with conserved current of Sect. 3.3.

Conserved charge:

$$Q = \int d^3x j^0(x) = iq \int d^3x \phi^* \overleftrightarrow{\partial}^0 \phi = iq \int d^3x (\pi^* \phi^* - \phi \pi), \quad (4.65)$$

with explicit solution (3.3) of free KG equation

$$\begin{aligned} Q &= -q \int d\vec{p} \frac{1}{2} \left[b^*(\vec{p})b(\vec{p}) - a^*(\vec{p})a(\vec{p}) - b(\vec{p})a(-\vec{p})e^{-2ip^0t} + a^*(\vec{p})b^*(-\vec{p})e^{2ip^0t} \right] + \text{c.c.} \\ &= q \int d\vec{p} (|a(\vec{p})|^2 - |b(\vec{p})|^2). \end{aligned} \quad (4.66)$$

⇒ Positive- and negative-frequency modes carry opposite charges,
in line with the antiparticle interpretation of the negative-frequency modes b !

- $SU(N)$ symmetry of N complex scalars:

n Noether currents:

$$\begin{aligned} j^{a,\mu} &= \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_k)} \Delta_k^a(\phi) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_k^*)} \Delta_k^a(\phi^*) \\ &= -ig [(\partial^\mu \phi_k^*) T_{kl}^a \phi_l - (\partial^\mu \phi_k) T_{kl}^a \phi_l^*], \quad n = 1, \dots, n. \end{aligned} \quad (4.67)$$

4.4.4 Translation invariance and energy-momentum tensor

Recall: space-time translations of fields and \mathcal{L} :

$$\Delta_k^\nu(\phi) = -\partial^\nu \phi_k, \quad \delta \mathcal{L} = -\delta \omega_\nu \partial_\mu K^{\mu\nu} \quad \text{with} \quad K^{\mu\nu} = -g^{\mu\nu} \mathcal{L}. \quad (4.68)$$

\Rightarrow “Current” $j \hat{=}$ energy-momentum tensor θ : (sign of prefactor = convention)

$$\theta^{\mu\nu} \equiv - \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_k)} \Delta_k^\nu(\phi) - K^{\mu\nu}(\phi) \right] = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_k)} \partial^\nu \phi_k - g^{\mu\nu} \mathcal{L}. \quad (4.69)$$

\Rightarrow Conserved “charges” Q form a 4-vector:

$$P^\mu \equiv \int d^3x \theta^{0\mu} = \int d^3x [\pi_k \partial^\mu \phi_k - g^{0\mu} \mathcal{L}]. \quad (4.70)$$

Individual components:

$$P^0 = \int d^3x [\pi_k \dot{\phi}_k - \mathcal{L}] = \int d^3x \mathcal{H} = H = \text{Hamilton function}, \quad (4.71)$$

$$\vec{P} = - \int d^3x \pi_k \vec{\nabla} \phi_k = \text{field momentum}. \quad (4.72)$$

\Rightarrow “Charge” conservation $\hat{=}$ conservation of energy and momentum of the fields.

$$\dot{P}^\mu = 0. \quad (4.73)$$

||| Comment:

The “current” of Lorentz transformations forms a rank-3 tensor, and the associated “charge” is the fields’ angular momentum.

Example: free complex scalar field

$$\mathcal{L} = (\partial_\mu \phi^*)(\partial^\mu \phi) - m^2 \phi^* \phi.$$

- Energy-momentum tensor:

$$\theta^{\mu\nu} = (\partial^\mu \phi^*)(\partial^\nu \phi) + (\partial^\nu \phi^*)(\partial^\mu \phi) - g^{\mu\nu} \mathcal{L}. \quad (4.74)$$

- Hamiltonian:

$$\mathcal{H} = \pi\dot{\phi} + \pi^*\dot{\phi}^* - \mathcal{L} = |\pi|^2 + |\vec{\nabla}\phi|^2 + m^2|\phi|^2, \quad P^0 = \int d^3x \mathcal{H}. \quad (4.75)$$

- Field momentum:

$$P^i = \int d^3x \theta^{0i} = \int d^3x (\dot{\phi}^* \partial^i \phi + \dot{\phi} \partial^i \phi^*). \quad (4.76)$$

- Insertion of plane-wave solutions yields

$$P^\mu = \int d\tilde{p} p^\mu (|a(\vec{p})|^2 + |b(\vec{p})|^2), \quad (4.77)$$

i.e. each mode characterized by $a(\vec{p})$, $b(\vec{p})$ carries energy $\propto p^0 = \sqrt{\vec{p}^2 + m^2}$, momentum $\propto \vec{p}$.

Non-uniqueness issue of $\theta^{\mu\nu}$

Possible redefinition of $\theta^{\mu\nu}$:

$$\tilde{\theta}^{\mu\nu} = \theta^{\mu\nu} + \partial_\rho \Sigma^{\rho\mu\nu}, \quad (4.78)$$

with any rank-3 tensor $\Sigma^{\rho\mu\nu} = -\Sigma^{\mu\rho\nu}$ (antisymmetry in the first two indices).

Features of $\tilde{\theta}^{\mu\nu}$:

- Conservation:

$$\partial_\mu \tilde{\theta}^{\mu\nu} = \underbrace{\partial_\mu \theta^{\mu\nu}}_{=0} + \underbrace{\partial_\mu \partial_\rho \Sigma^{\rho\mu\nu}}_{=0 \text{ by symmetry}} = 0. \quad (4.79)$$

- Field momentum:

$$\begin{aligned} \tilde{P}^\mu &\equiv \int_V d^3x \tilde{\theta}^{0\mu} = P^\mu + \int_V d^3x \underbrace{\partial_\rho \Sigma^{\rho 0\mu}}_{= \partial_i \Sigma^{i0\mu}, \text{ since } \Sigma^{00\mu} = 0} \\ &\stackrel{\text{Gauss}}{=} P^\mu + \underbrace{\oint_{A(V)} d^2A^i \Sigma^{i0\mu}}_{=0 \text{ if } \Sigma \text{ vanishes fast enough for } |x^\mu| \rightarrow \infty} = P^\mu, \end{aligned} \quad (4.80)$$

\Rightarrow Both tensors are equally suited as energy-momentum tensors.

||| Comment:

||| This freedom can be used to construct conserved tensors with desired properties (e.g. symmetry, gauge invariance) that are not automatically satisfied by the form directly obtained from the Noether theorem.

Chapter 5

Canonical quantization of free scalar fields

5.1 Canonical commutation relations

Consider discrete system with coordinates $q_k(t)$ and canonical conjugate momenta $p_k(t)$ and its quantization in the *Heisenberg picture*:

$q_k(t), p_k(t)$ are hermitian operators obeying

- the classical EOMs;
- Heisenberg's commutation relations.

↔ Take continuum limit $q_k(t) \rightarrow \phi(t, \vec{x})$!

Comment:

Heisenberg picture is more appropriate for field quantization than Schrödinger picture, because the EOM for the field (which becomes an operator) is known. Recall the connection of the two pictures by the unitary transformation for time evolution:

$$\underbrace{|\psi\rangle_{\text{H}}} \equiv |\psi(t_0)\rangle = U^{-1}(t, t_0) \underbrace{|\psi(t)\rangle}_{\substack{\text{qm. state in S picture} \\ = t\text{-dependent}}} = \text{const.} \quad \text{for some fixed } t_0, \quad (5.1)$$

$$\underbrace{\hat{O}_{\text{H}}(t)}_{\substack{\text{qm. operator} \\ \text{in H picture}}} \equiv U^{-1}(t, t_0) \underbrace{\hat{O}}_{\substack{\text{qm. operator} \\ \text{in S picture}}} U(t, t_0), \quad (5.2)$$

where the time evolution operator U satisfies the differential equation

$$i \frac{dU(t, t_0)}{dt} = \hat{H}U(t, t_0) \quad \text{with} \quad U(t_0, t_0) = \mathbf{1}. \quad (5.3)$$

Quantization procedure:

Discrete system:

- Canonical variables $q_k(t)$, $p_k(t)$ obey commutators:

$$[q_k(t), p_l(t)] = i\hbar\delta_{kl},$$

$$[q_k(t), q_l(t)] = 0,$$

$$[p_k(t), p_l(t)] = 0$$

Note: The commutator relations only hold for equal times.

- H and L are hermitian operators obtained from classical quantities via the correspondence principle:

$$H(q_k^{\text{cl}}, p_l^{\text{cl}}) \xrightarrow[\text{reordering}]{\text{unique up to}} H(q_k, p_l)$$

- The operators fulfill the EOMs:

$$\frac{dq_k(t)}{dt} = \frac{1}{i\hbar} [q_k(t), H],$$

$$\frac{dp_k(t)}{dt} = \frac{1}{i\hbar} [p_k(t), H]$$

($\hat{=}$ classical EOM: $\frac{1}{i\hbar}[\cdot, \cdot] \hat{=} \{\cdot, \cdot\}$)

- States $|\Psi\rangle$ are time independent.

Continuous system:

- Canonical field operators $\phi_k(x)$, $\pi_k(x)$ obey

$$[\phi_k(t, \vec{x}), \pi_l(t, \vec{y})] = i\hbar\delta_{kl}\delta(\vec{x} - \vec{y}),$$

$$[\phi_k(t, \vec{x}), \phi_l(t, \vec{y})] = 0,$$

$$[\pi_k(t, \vec{x}), \pi_l(t, \vec{y})] = 0$$

Note: The commutators only hold for equal times $t = x^0 = y^0$.

- The hermitian operators $\mathcal{H} = \pi\dot{\phi} - \mathcal{L}$ and \mathcal{L} are obtained analogously:

$$\mathcal{L}(\phi_k^{\text{cl}}, \partial\phi_k^{\text{cl}}) \rightarrow \mathcal{L}(\phi_k, \partial\phi_k)$$

- EOMs:

$$\frac{\partial\phi_k(t)}{\partial t} = \frac{1}{i\hbar} [\phi_k(t), H],$$

$$\frac{\partial\pi_k(t)}{\partial t} = \frac{1}{i\hbar} [\pi_k(t), H]$$

- States $|\Psi\rangle$ are time independent.

- Microcausality for space-like distances is demanded:

$$[\phi_k(x), \pi_l(y)] = 0,$$

$$[\phi_k(x), \phi_l(y)] = 0, \text{ etc.,}$$

for $(x - y)^2 < 0$, also for $x^0 \neq y^0$.

Comment:

In the canonical formalism Lorentz covariance is not manifest as

- ★ the commutator relations are imposed at equal times,
- ★ the Hamilton operator is not a Lorentz scalar,
- ★ time is singled out in the EOMs.

Observables are, however, Lorentz invariant which can be shown, for example, via the functional integral formalism.

5.2 Free Klein–Gordon field

Classical:

Real KG field $\phi(x)$

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2$$

$$\Rightarrow (\square + m^2)\phi = 0$$

Complex KG fields $\phi(x), \phi(x)^*$

$$\mathcal{L} = (\partial_\mu\phi^*)(\partial^\mu\phi) - m^2\phi^*\phi$$

QFT:

→ hermitian field operator $\phi^\dagger(x) = \phi(x)$

$$\rightarrow \mathcal{L} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2$$

$$\Rightarrow (\square + m^2)\phi = 0 \quad (\text{operator equation})$$

→ non-hermitian field operators $\phi(x), \phi^\dagger(x)$

$$\rightarrow \mathcal{L} = (\partial_\mu\phi^\dagger)(\partial^\mu\phi) - m^2\phi^\dagger\phi$$

Solution of KG equation:

$$\phi(x) = \int d\tilde{p} [a(\vec{p})e^{-ipx} + b^*(\vec{p})e^{ipx}] \quad \rightarrow \quad \phi(x) = \int d\tilde{p} [a(\vec{p})e^{-ipx} + b^\dagger(\vec{p})e^{ipx}]$$

functions $a(\vec{p}), b(\vec{p})$

→ operators $a(\vec{p}), b(\vec{p})$

real case: $b(\vec{p}) = a(\vec{p})$

real case: $b(\vec{p}) = a(\vec{p})$

Canonical momentum:

$$\pi = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \dot{\phi}^*, \quad \pi^* = \frac{\partial\mathcal{L}}{\partial\dot{\phi}^*} = \dot{\phi} \quad \rightarrow \quad \pi = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \dot{\phi}^\dagger, \quad \pi^\dagger = \frac{\partial\mathcal{L}}{\partial\dot{\phi}^\dagger} = \dot{\phi}$$

Meaning of the operators $a, a^\dagger, b, b^\dagger$?

1. Calculate $a, a^\dagger, b, b^\dagger$ from $\phi(x), \phi^\dagger(x)$ (inverse Fourier transformation):

$$\begin{aligned} \int d^3x e^{-i\vec{q}\cdot\vec{x}} \phi(x) &= \int d\tilde{p} \left[a(\vec{p})e^{-ip^0x^0} (2\pi)^3 \delta(\vec{p} - \vec{q}) + b^\dagger(\vec{p})e^{ip^0x^0} (2\pi)^3 \delta(\vec{p} + \vec{q}) \right] \\ &= \frac{1}{2q^0} \left[a(\vec{q})e^{-iq^0x^0} + b^\dagger(-\vec{q})e^{iq^0x^0} \right]_{q^0=\sqrt{\vec{q}^2+m^2}}, \end{aligned} \quad (5.4)$$

$$\int d^3x e^{-i\vec{q}\cdot\vec{x}} \dot{\phi}(x) = -\frac{i}{2} \left[a(\vec{q})e^{-iq^0x^0} - b^\dagger(-\vec{q})e^{iq^0x^0} \right]_{q^0=\sqrt{\vec{q}^2+m^2}}. \quad (5.5)$$

$$\Rightarrow a(\vec{q}) = i \int d^3x \underbrace{e^{iq^0x^0} e^{-i\vec{q}\cdot\vec{x}}}_{=e^{iqx}} [-iq^0\phi(x) + \dot{\phi}(x)] = i \int d^3x e^{iqx} \overleftrightarrow{\partial}_0 \phi(x), \quad (5.6)$$

$$b(\vec{q}) = i \int d^3x e^{iqx} \overleftrightarrow{\partial}_0 \phi^\dagger(x) \quad (\text{derived analogously}) \quad (5.7)$$

with

$$f \overleftrightarrow{\partial}_\mu g = f(\partial_\mu g) - (\partial_\mu f)g. \quad (5.8)$$

2. Commutator relations: (choose $x^0 = y^0$)

$$\begin{aligned} \bullet [a(\vec{q}), a^\dagger(\vec{p})] &= \int d^3x \int d^3y e^{iqx} e^{-ipy} \underbrace{[-iq^0\phi(x) + \dot{\phi}(x), ip^0\phi^\dagger(y) + \dot{\phi}^\dagger(y)]}_{\substack{x^0=y^0 \\ q^0=\sqrt{\vec{q}^2+m^2} \\ p^0=\sqrt{\vec{p}^2+m^2}}} \\ &= [-iq^0\phi(x) + \pi^\dagger(x), ip^0\phi^\dagger(y) + \pi(y)] \\ &= -iq^0[\phi(x), \pi(y)] + ip^0[\pi^\dagger(x), \phi^\dagger(y)] \\ &= -iq^0i\delta(\vec{x}-\vec{y}) + ip^0(-i)\delta(\vec{x}-\vec{y}) \\ &= \int d^3x e^{i(q-p)x} (q^0 + p^0) \Big|_{\substack{q^0=\sqrt{\vec{q}^2+m^2} \\ p^0=\sqrt{\vec{p}^2+m^2}}} \\ &= (2\pi)^3 2\sqrt{\vec{p}^2+m^2} \delta(\vec{q}-\vec{p}), \end{aligned} \quad (5.9)$$

$$\bullet [b(\vec{q}), b^\dagger(\vec{p})] = \dots = (2\pi)^3 2\sqrt{\vec{p}^2+m^2} \delta(\vec{q}-\vec{p}), \quad (5.10)$$

$$\bullet [a(\vec{q}), a(\vec{p})] = \dots = [b^\dagger(\vec{q}), b^\dagger(\vec{p})] = 0 \quad \text{for all other commutators.} \quad (5.11)$$

3. Energy and momentum 4-vector:

$$P^\mu = \int d^3x [\pi(\partial^\mu\phi) + \pi^\dagger(\partial^\mu\phi^\dagger) - g^{\mu 0}\mathcal{L}] \quad (5.12)$$

|| Comment:
|| Ordering issue of operators solved later (*normal ordering*).

Energy:

$$H = P^0 = \int d^3x [2\pi^\dagger\pi - (\partial_\mu\phi^\dagger)\partial^\mu\phi + m^2\phi^\dagger\phi] \quad (5.13)$$

$$= \int d^3x [\pi^\dagger\pi + (\nabla\phi^\dagger)(\nabla\phi) + m^2\phi^\dagger\phi] \quad (5.14)$$

$$= \dots = \int d\vec{p} \frac{1}{2} p^0 [a(\vec{p})a^\dagger(\vec{p}) + a^\dagger(\vec{p})a(\vec{p}) + b(\vec{p})b^\dagger(\vec{p}) + b^\dagger(\vec{p})b(\vec{p})]. \quad (5.15)$$

3-momentum:

$$\begin{aligned}\vec{P} &= - \int d^3x \left[\pi \nabla \phi + \pi^\dagger \nabla \phi^\dagger \right] \\ &= \dots = \int d\tilde{p} \frac{1}{2} \vec{p} \left[a(\vec{p}) a^\dagger(\vec{p}) + a^\dagger(\vec{p}) a(\vec{p}) + b(\vec{p}) b^\dagger(\vec{p}) + b^\dagger(\vec{p}) b(\vec{p}) \right].\end{aligned}\quad (5.16)$$

Commutation relations:

$$\begin{aligned}[H, a^\dagger(\vec{p})] &= \frac{1}{2} \int d\tilde{q} q^0 \left\{ \underbrace{[a(\vec{q}), a^\dagger(\vec{p})]}_{=(2\pi)^3 2p^0 \delta(\vec{q}-\vec{p})} a^\dagger(\vec{q}) + a^\dagger(\vec{q}) \underbrace{[a(\vec{q}), a^\dagger(\vec{p})]}_{=(2\pi)^3 2p^0 \delta(\vec{q}-\vec{p})} \right\} \\ &= p^0 a^\dagger(\vec{p}),\end{aligned}\quad (5.17)$$

$$[H, a(\vec{p})] = - [H, a^\dagger(\vec{p})]^\dagger = -p^0 a(\vec{p}), \quad (5.18)$$

$$[\vec{P}, a^\dagger(\vec{p})] = \vec{p} a^\dagger(\vec{p}), \quad [\vec{P}, a(\vec{p})] = -\vec{p} a(\vec{p}), \quad (5.19)$$

$$\Rightarrow [P^\mu, a^\dagger(\vec{p})] = p^\mu a^\dagger(\vec{p}), \quad [P^\mu, a(\vec{p})] = -p^\mu a(\vec{p}), \quad (5.20)$$

$$[P^\mu, b^\dagger(\vec{p})] = p^\mu b^\dagger(\vec{p}), \quad [P^\mu, b(\vec{p})] = -p^\mu b(\vec{p}). \quad (\text{derived analogously}) \quad (5.21)$$

4. Comparison with system of independent harmonic oscillators of quantum mechanics:

$$H = \sum_k \left[\frac{p_k^2}{2m} + \frac{1}{2} m \omega^2 q_k^2 \right] = \sum_k \frac{\hbar \omega}{2} (a_k a_k^\dagger + a_k^\dagger a_k)$$

with shift operators a_k, a_k^\dagger obeying

$$[a_k, a_l^\dagger] = \delta_{kl}, \quad [a_k, a_l] = [a_k^\dagger, a_l^\dagger] = 0 \quad [H, a_k^\dagger] = \hbar \omega a_k^\dagger, \quad [H, a_k] = -\hbar \omega a_k.$$

Illustration for energy eigenstate $|E\rangle$:

$$H \left(a_k^\dagger |E\rangle \right) = [H, a_k^\dagger] |E\rangle + a_k^\dagger H |E\rangle = (\hbar \omega + E) a_k^\dagger |E\rangle,$$

i.e. $a_k^\dagger |E\rangle$ is energy eigenstate to energy $E + \hbar \omega$ (a_k^\dagger “creates” energy $\hbar \omega$).

\Rightarrow Interpretation of $a(\vec{p}), a^\dagger(\vec{p}), b(\vec{p}), b^\dagger(\vec{p})$ as creation and annihilation operators for field modes (=particle excitations):

- $a^{(\dagger)}$ and $b^{(\dagger)}$ correspond to two independent, free particle types X and \bar{X} , respectively, both with mass m .
- $a(\vec{p}) / a^\dagger(\vec{p})$ annihilates / creates particle X with energy $\hbar \omega = p^0 = \sqrt{\vec{p}^2 + m^2}$ and 3-momentum \vec{p} (de Broglie momentum).

- $b(\vec{p})$ / $b^\dagger(\vec{p})$ annihilates / creates particle \bar{X} with energy $\hbar\omega = p^0 = \sqrt{\vec{p}^2 + m^2}$ and 3-momentum \vec{p} .

5. Electric current density and charge operators [cf. Eq. (4.64)]:

$$j^\mu = -iq [(\partial^\mu \phi^\dagger)\phi - \phi^\dagger(\partial^\mu \phi)] = iq\phi^\dagger \overleftrightarrow{\partial}^\mu \phi, \quad (5.22)$$

$$Q = iq \int d^3x (\phi^\dagger \pi^\dagger - \pi \phi) = iq\phi^\dagger \overleftrightarrow{\partial}^0 \phi \quad (5.23)$$

$$= \dots = q \int d\vec{p} [a^\dagger(\vec{p})a(\vec{p}) - b(\vec{p})b^\dagger(\vec{p})]. \quad (5.24)$$

|| Comment:
|| Ordering issue of operators solved later (*normal ordering*).

⇒ Commutation relations:

$$[Q, a^\dagger(\vec{p})] = +qa^\dagger(\vec{p}), \quad [Q, a(\vec{p})] = -qa(\vec{p}), \quad (5.25)$$

$$[Q, b^\dagger(\vec{p})] = -qb^\dagger(\vec{p}), \quad [Q, b(\vec{p})] = +qb(\vec{p}), \quad (5.26)$$

i.e. a^\dagger and b increase charge by amount q , while a and b^\dagger reduce charge by amount q .

⇒ Particle X carries charge $+q$, particle \bar{X} carries charge $-q$ ($\bar{X} = \text{antiparticle}$).

Def.: *Charge conjugation* C

$$\phi^C(x) \equiv \phi^\dagger(x) = \int d\vec{p} [b(\vec{p})e^{-ipx} + a^\dagger(\vec{p})e^{ipx}], \quad (\phi^\dagger)^C(x) \equiv \phi(x), \quad (5.27)$$

i.e. C interchanges particle and antiparticle.

Real KG field:

- Hermitian field operator $\phi(x) = \phi^\dagger(x) = \phi^C(x)$, i.e. $a(\vec{p}) = b(\vec{p})$.
⇒ $X \equiv \bar{X}$ (Particle is its own antiparticle.)
- Factor 1/2 in \mathcal{H} and \mathcal{L} .

$$\Rightarrow P^\mu = \int d\vec{p} \frac{1}{2} p^\mu [a(\vec{p})a^\dagger(\vec{p}) + a^\dagger(\vec{p})a(\vec{p})] \quad (5.28)$$

with

$$[P^\mu, a^\dagger(\vec{p})] = p^\mu a^\dagger(\vec{p}), \quad [P^\mu, a(\vec{p})] = -p^\mu a(\vec{p}). \quad (5.29)$$

- Electric current and charge operators: $j^\mu = \text{const.} \times \mathbf{1}$, $Q = \text{const.} \times \mathbf{1}$.

$$\Rightarrow [Q, a^\dagger(\vec{p})] = [Q, a(\vec{p})] = 0, \quad (5.30)$$

i.e. particle creation / annihilation does not change overall charge.

⇒ Charge $q_X = 0$, X is electrically neutral.

|| Comment:
 || The problem with the (divergent) constant in Q is solved by *normal ordering* (=part
 || of *renormalization process*).

5.3 Particle states and Fock space

Idea: construct Hilbert space of qm. states upon applying creation operators to ground state (analogy to qm. harmonic oscillator).

Definition: *Fock space*

- Ground state $|0\rangle$ (*vacuum*, no particle excitation):

$$|0\rangle : \quad a(\vec{p})|0\rangle = 0, \quad b(\vec{p})|0\rangle = 0 \quad \forall \vec{p}, \quad (5.31)$$

$$\langle 0| = (|0\rangle)^\dagger : \quad \langle 0|a^\dagger(\vec{p}) = 0, \quad \langle 0|b^\dagger(\vec{p}) = 0, \quad (5.32)$$

$$\text{Normalization:} \quad \langle 0|0\rangle = 1. \quad (5.33)$$

Note: $|0\rangle$ exists, otherwise energy is not bounded from below.

- Excited states (particle states):

$$|X(\vec{p}_1)\rangle = a^\dagger(\vec{p}_1)|0\rangle \quad 1 \text{ particle} \quad (5.34)$$

$$|\bar{X}(\vec{p}_1)\rangle = b^\dagger(\vec{p}_1)|0\rangle \quad 1 \text{ antiparticle} \quad (5.35)$$

$$|X(\vec{p}_1)X(\vec{p}_2)\rangle = a^\dagger(\vec{p}_1)a^\dagger(\vec{p}_2)|0\rangle \quad 2 \text{ particles} \quad (5.36)$$

$$|\bar{X}(\vec{p}_1)\bar{X}(\vec{p}_2)\rangle = b^\dagger(\vec{p}_1)b^\dagger(\vec{p}_2)|0\rangle \quad 2 \text{ antiparticles} \quad (5.37)$$

$$|X(\vec{p}_1)\bar{X}(\vec{p}_2)\rangle = a^\dagger(\vec{p}_1)b^\dagger(\vec{p}_2)|0\rangle \quad 1 \text{ particle, 1 antiparticle}$$

$$\vdots \quad (5.38)$$

$$|X(\vec{p}_1)\dots X(\vec{p}_n)\rangle = a^\dagger(\vec{p}_1)\dots a^\dagger(\vec{p}_n)|0\rangle \quad n \text{ particles} \quad (5.39)$$

⋮

- *Fock space* = Hilbert space spanned by all the particle and antiparticle states:

$$\{ |0\rangle, |X(\vec{p}_1)\rangle, |\bar{X}(\vec{q}_1)\rangle, \dots, |X(\vec{p}_1)\dots X(\vec{p}_n)\bar{X}(\vec{q}_1)\dots \bar{X}(\vec{q}_m)\rangle, \dots \}$$

Properties of Fock states:

- The many-particle states are symmetric with respect to particle exchange:

$$\begin{aligned} |\dots X(\vec{p}_i) \dots X(\vec{p}_j) \dots\rangle &= |\dots X(\vec{p}_j) \dots X(\vec{p}_i) \dots\rangle \\ |\dots \bar{X}(\vec{p}_i) \dots \bar{X}(\vec{p}_j) \dots\rangle &= |\dots \bar{X}(\vec{p}_j) \dots \bar{X}(\vec{p}_i) \dots\rangle \end{aligned} \quad (5.40)$$

\Rightarrow Particles X and \bar{X} are *bosons*. (Fermionic states are antisymmetric.)

- Normalization of one-particle states:

$$\begin{aligned} \langle X(\vec{p}) | X(\vec{q}) \rangle &= \langle 0 | a(\vec{p}) a^\dagger(\vec{q}) | 0 \rangle \stackrel{a|0\rangle=0}{=} \langle 0 | [a(\vec{p}), a^\dagger(\vec{q})] | 0 \rangle \\ &= (2\pi)^3 2p^0 \delta(\vec{p} - \vec{q}) \underbrace{\langle 0 | 0 \rangle}_{=1} = (2\pi)^3 2p^0 \delta(\vec{p} - \vec{q}) = \text{Lorentz invariant}, \end{aligned} \quad (5.41)$$

$$\langle \bar{X}(\vec{p}) | \bar{X}(\vec{q}) \rangle = \dots = (2\pi)^3 2p^0 \delta(\vec{p} - \vec{q}), \quad (5.42)$$

$$\langle X(\vec{p}) | \bar{X}(\vec{q}) \rangle = 0. \quad (5.43)$$

Note: $\langle X(\vec{p}) | X(\vec{p}) \rangle \rightarrow \infty$ for momentum eigenstates.

\hookrightarrow Wave packets needed to obtain normalized one-particle states $|\Psi\rangle$ with $\langle \Psi | \Psi \rangle = 1$.

Vacuum state and normal ordering:

Vacuum state $|0\rangle$: no particle, i.e. $\langle 0 | P^\mu | 0 \rangle \stackrel{!}{=} 0$, $\langle 0 | Q | 0 \rangle \stackrel{!}{=} 0$, etc.

But: These conditions are not fulfilled automatically.

\hookrightarrow Enforce condition “by hand” (concept of *normal ordering*), since usually only changes in energy, momentum, etc., are measurable.

Example: vacuum energy of real scalar field

$$\begin{aligned} \langle 0 | H | 0 \rangle &= \langle 0 | \int d\vec{p} \frac{1}{2} p^0 [a(\vec{p}) a^\dagger(\vec{p}) + a^\dagger(\vec{p}) a(\vec{p})] | 0 \rangle \\ &= \int d\vec{p} \frac{p^0}{2} \left[\underbrace{\langle 0 | [a(\vec{p}), a^\dagger(\vec{p})] | 0 \rangle}_{=(2\pi)^3 2p^0 \delta(\vec{p} - \vec{p}),} + 2 \underbrace{\langle 0 | a^\dagger(\vec{p}) a(\vec{p}) | 0 \rangle}_{=0} \right] \\ &\quad \text{with } (2\pi)^3 \delta(\vec{p} - \vec{p}) \rightarrow V = \text{space volume} \\ &= V \underbrace{\int \frac{d^3 p}{(2\pi)^3}}_{\text{number of states in } V} \frac{p^0}{2} \rightarrow \infty \end{aligned} \quad (5.44)$$

$\Rightarrow \langle H \rangle$ for a state $|f\rangle = \int d\vec{p} f(\vec{p}) |X(\vec{p})\rangle$ (wave packet),

f being a square-integrable function, $\int d\vec{p} |f(\vec{p})|^2 < \infty$:

$$\langle f|H|f\rangle = \underbrace{\int d\vec{p} |f(\vec{p})|^2 p^0}_{\text{finite, observable}} + \underbrace{\langle 0|H|0\rangle}_{\substack{\rightarrow \infty, \\ \text{non-observable for} \\ \text{time-independent } |0\rangle}} \quad (5.45)$$

\Leftrightarrow Redefinition of H upon subtracting $\langle 0|H|0\rangle$

Definition: *normal ordering*

$$: A : \equiv A \Big|_{\text{all annihilation operators shifted to the right}} \Rightarrow \langle 0 | : A : | 0 \rangle = 0 \quad (5.46)$$

Examples: $: a(\vec{p}) a^\dagger(\vec{p}) : = a^\dagger(\vec{p}) a(\vec{p})$,
 $: a(\vec{k}) a^\dagger(\vec{k}) a(\vec{p}) a^\dagger(\vec{p}) : = a^\dagger(\vec{k}) a^\dagger(\vec{p}) a(\vec{k}) a(\vec{p})$.

Redefinition of all operators, also of \mathcal{L} , \mathcal{H} , etc.:

- Definitions:

$$\text{particle number density operator:} \quad N_X(\vec{p}) = a^\dagger(\vec{p}) a(\vec{p}), \quad (5.47)$$

$$\text{particle number operator:} \quad N_X = \int d\vec{p} N_X(\vec{p}), \quad (5.48)$$

$$\text{antiparticle number density operator:} \quad N_{\bar{X}}(\vec{p}) = b^\dagger(\vec{p}) b(\vec{p}), \quad (5.49)$$

$$\text{antiparticle number operator:} \quad N_{\bar{X}} = \int d\vec{p} N_{\bar{X}}(\vec{p}). \quad (5.50)$$

- 4-momentum operator:

$$P^\mu = \int d\vec{p} p^\mu [N_X(\vec{p}) + N_{\bar{X}}(\vec{p})]. \quad (5.51)$$

- electric charge operator:

$$Q = q \int d\vec{p} [N_X(\vec{p}) - N_{\bar{X}}(\vec{p})]. \quad (5.52)$$

5.4 Field operator and wave function

Interpretation of field operator

Consider one-particle wave packet for particle X (charged scalar):

$$|f\rangle = \int d\vec{p} f(\vec{p}) |X(\vec{p})\rangle = \int d\vec{p} f(\vec{p}) a^\dagger(\vec{p}) |0\rangle \quad (5.53)$$

Interference with state $\phi^\dagger(x) |0\rangle$:

$$\langle 0 | \phi(x) | f \rangle = \int d\vec{p} f(\vec{p}) \langle 0 | \phi(x) a^\dagger(\vec{p}) | 0 \rangle \quad (5.54)$$

$$= \int d\vec{p} f(\vec{p}) \int d\vec{k} e^{-i\vec{k}x} \underbrace{\langle 0 | a(\vec{k}) a^\dagger(\vec{p}) | 0 \rangle}_{= (2\pi)^3 (2k^0) \delta(\vec{k}-\vec{p})} \quad (5.55)$$

$$= \int d\vec{p} e^{-i\vec{p}x} f(\vec{p}) \equiv \varphi_f(x). \quad (5.56)$$

Compare with wave packet of non-relat. QM:

$$|\psi_f\rangle = |\psi_f(0)\rangle = \int \frac{d^3p}{(2\pi)^3} f(\vec{p}) |\vec{p}\rangle,$$

$$|\psi_f(t)\rangle = U(t, 0) |\psi_f(0)\rangle = \exp\{-iHt\} |\psi_f\rangle = \int \frac{d^3p}{(2\pi)^3} f(\vec{p}) e^{-ip^0t} |\vec{p}\rangle \Big|_{p^0 = \frac{\vec{p}^2}{2m}},$$

$$\begin{aligned} \psi_f(t, \vec{x}) &= \langle \vec{x} | \psi_f(t) \rangle = \int \frac{d^3p}{(2\pi)^3} f(\vec{p}) e^{-ip^0t} \underbrace{\langle \vec{x} | \vec{p} \rangle}_{= e^{i\vec{p}\vec{x}}} = \int \frac{d^3p}{(2\pi)^3} e^{-ipx} f(\vec{p}) \\ &= \langle \vec{x} | \exp\{-iHt\} | \psi_f \rangle \end{aligned} \quad (5.57)$$

$\Rightarrow \varphi(x)$ is analogue of one-particle wave function $\psi(t, \vec{x}) = \langle \vec{x} | \psi(t) \rangle$ in QM.

$\phi^\dagger(x) |0\rangle$ is analogue of $\exp\{+iHt\} |\vec{x}\rangle =$ Heisenberg state ($t = 0$) corresponding to position eigenstate $|\vec{x}\rangle$ at time t , i.e. describes particle created at position \vec{x} at time t .

Space-time transformations: $x \rightarrow x' = \Lambda x + a$

- Qm. states:

$$|f\rangle \rightarrow |f'\rangle = U(\Lambda, a) |f\rangle \quad \text{with } U = \text{unitary operator.} \quad (5.58)$$

\Leftrightarrow Transition amplitudes $\langle f' | g' \rangle = \langle f | U^\dagger U | g \rangle = \langle f | g \rangle =$ invariant.

- Field operator:

$$\phi(x') = U(\Lambda, a) \phi(x) U^\dagger(\Lambda, a), \quad (5.59)$$

so that scalar products $\langle f | \dots \phi(x) \dots | g \rangle = \langle f' | \dots \phi(x') \dots | g' \rangle =$ invariant.

- Wave function:

$$\begin{aligned}\varphi'(x') &= \langle 0 | \phi(x') | f' \rangle = \underbrace{\langle 0 | U(\Lambda, a)}_{= \langle 0 | = \text{invariant}} \phi(x) \underbrace{U^\dagger(\Lambda, a) U(\Lambda, a)}_{= \mathbf{1}} | f \rangle \\ &= \langle 0 | \phi(x) | f \rangle = \varphi(x).\end{aligned}\tag{5.60}$$

$$\Rightarrow \varphi'(x) = \varphi(\Lambda^{-1}(x - a)).\tag{5.61}$$

5.5 Propagator and time ordering

Concept of *time ordering* of operators is very important in QFT.

Definition: *Time-ordering operator* T

$$T[\phi_1(x_1) \dots \phi_n(x_n)] \equiv \phi_{i_1}(x_{i_1}) \phi_{i_2}(x_{i_2}) \dots \phi_{i_n}(x_{i_n}) \quad \text{for } x_{i_1}^0 > x_{i_2}^0 > \dots > x_{i_n}^0,\tag{5.62}$$

i.e. “operators with earlier times are applied first”.

(Feynman) Propagator of complex scalar field (cf. Sect. 4.3.2)

$$iD_F(x, y) = \langle 0 | T [\phi(x) \phi^\dagger(y)] | 0 \rangle.\tag{5.63}$$

Proof:

Time-ordered product of two fields:

$$\begin{aligned}T[\phi(x) \phi^\dagger(y)] &= \phi(x) \phi^\dagger(y) \theta(x^0 - y^0) + \phi^\dagger(y) \phi(x) \theta(y^0 - x^0).\tag{5.64} \\ \Rightarrow \langle 0 | T [\phi(x) \phi^\dagger(y)] | 0 \rangle &= \langle 0 | T \int d\tilde{p} [a(\tilde{p}) e^{-ipx} + b^\dagger(\tilde{p}) e^{ipx}] \\ &\quad \times \int d\tilde{q} [a^\dagger(\tilde{q}) e^{iqy} + b(\tilde{q}) e^{-iqy}] | 0 \rangle \\ &= \theta(x^0 - y^0) \int d\tilde{p} \int d\tilde{q} e^{-i(px - qy)} \langle 0 | a(\tilde{p}) a^\dagger(\tilde{q}) | 0 \rangle \\ &\quad + \theta(y^0 - x^0) \int d\tilde{p} \int d\tilde{q} e^{i(px - qy)} \langle 0 | b(\tilde{q}) b^\dagger(\tilde{p}) | 0 \rangle\end{aligned}$$

- $x^0 > y^0$: particle creation at y , propagation to x , annihilation at x
- $x^0 < y^0$: antiparticle creation at x , propagation to y , annihilation at y
- Note: identical charge flow from y to x in both cases

$$= \theta(x^0 - y^0) \int d\tilde{p} e^{-ip(x-y)} + \theta(y^0 - x^0) \int d\tilde{p} e^{ip(x-y)}$$

$$\begin{aligned} &= \int \frac{d^4p}{(2\pi)^4} \frac{e^{-ip(x-y)}}{p^2 - m^2 + i\epsilon} \quad [\text{see Eq. (4.42)}] \\ &= iD_F(x, y) \end{aligned} \tag{5.65}$$

q.e.d.

Chapter 6

Interacting scalar fields and scattering theory

QFT with interacting fields

↔ framework for formulating theories of fundamental interactions

But: evaluation extremely complicated

↔ systematic approximative methods needed

- Exact solutions: only for some lower-dimensional models
- *Perturbation theory*: expansion in small coupling constants g
↔ most useful for scattering problems
- *Lattice calculations*: numerical simulations by discretizing space–time
↔ useful for static problems (e.g. for bound states in strong interaction)

6.1 Asymptotic states and S-matrix

Asymptotic states in particle scattering

Scattering process:

$$|i\rangle \rightarrow |f\rangle \tag{6.1}$$

with

$|i\rangle$ = prepared momentum eigenstate before scattering,
evolving into a complicated mixed many-particle state $|i'\rangle$ after scattering,

$|f\rangle$ = specific final state that is contained in $|i'\rangle$ after scattering.

Relevant cases:

- 2-particle scattering: $|i\rangle = |\vec{k}_1, \vec{k}_2\rangle$
- particle decay: $|i\rangle = |\vec{k}_1\rangle$

Technical description:

- interaction in finite time interval $[-T, T]$ with $T \gg$ any relevant time scale
- initial state $|i\rangle_{\text{in}}$: $t \rightarrow -\infty$ ($t < -T$), no interaction
- final state $|f\rangle_{\text{out}}$: $t \rightarrow +\infty$ ($t > +T$), no interaction

Subtleties:

- Behaviour of $\phi(x)$ for $t = x^0 \rightarrow \mp\infty$:

$$\phi(x) \underset{t \rightarrow \mp\infty}{\sim} \underbrace{Z^{1/2}}_{\substack{\text{wave-function} \\ \text{renormalization} \\ \text{constant}}} \phi_{\text{in/out}}(x), \quad (6.2)$$

where the asymptotics holds in the “weak” sense (for matrix elements only).

Origin of Z :

$\phi(x)$ and “free” fields $\phi_{\text{in/out}}(x)$ are canonically normalized (commutators!), but

- Free fields only have non-vanishing matrix elements with one-particle states:

$$\langle 0 | \phi_{\text{in}}(x) | \vec{k} \rangle_{\text{in}} = e^{-ikx}. \quad (6.3)$$

- Interacting fields interfere also with multiparticle states:

$$\langle 0 | \phi(x) | \vec{k}_1 \dots \vec{k}_n \rangle \neq 0. \quad (6.4)$$

- Relation between wave functions:

$$\langle 0 | \phi(x) | \vec{k} \rangle = Z^{1/2} \langle 0 | \phi_{\text{in/out}}(x) | \vec{k} \rangle \quad (6.5)$$

|| Comment:
 || Z can be calculated from the vacuum expectation value of $[\phi(x), \phi(y)]$ (see e.g.
 || [2, 3]). Under some assumption (finiteness), one can show that $0 < Z < 1$.

- Interaction changes mass value (*mass renormalization*):
 Asymptotic fields satisfy the free KG equation,

$$(\square + \bar{m}^2)\phi_{\text{in/out}}(x) = 0, \quad (6.6)$$

but mass $\bar{m} \neq m =$ mass in original Lagrangian.

- Note: in lowest order of perturbation theory: $Z = 1$ and $\bar{m} = m$.
 Higher orders deserve care, i.e. a proper *renormalization*.

Scattering operator, “S-matrix”

Free in/out states form orthonormal bases (ONBs) of two isomorphic Fock spaces.

- “in” states:

$\{|\alpha\rangle_{\text{in}}\}$ = ONB of states with definite particle number,

$\{S|\alpha\rangle_{\text{in}}\}$ = ONB of free states after scattering (no definite particle number).

\Leftrightarrow Scattering operator S is unitary: $S^\dagger = S^{-1}$.

- “out” states:

$|\alpha\rangle_{\text{out}} \equiv S^\dagger |\alpha\rangle_{\text{in}}$ = free state with definite particle number *after scattering*
(because $S|\alpha\rangle_{\text{out}} = |\alpha\rangle_{\text{in}}$).

Vacuum states can be identified:

$$S|0\rangle_{\text{out}} = |0\rangle_{\text{in}} \equiv |0\rangle_{\text{out}} \equiv |0\rangle. \quad (6.7)$$

- Probabilities for qm. transitions $|i\rangle \rightarrow |f\rangle$ are proportional to $|S_{fi}|^2$ where

$$S_{fi} = {}_{\text{in}}\langle f|S|i\rangle_{\text{in}} = {}_{\text{out}}\langle f|i\rangle_{\text{in}} = {}_{\text{out}}\langle f|S|i\rangle_{\text{out}}, \quad (6.8)$$

$|i\rangle_{\text{in}}$ = prepared free momentum eigenstate,

$|f\rangle_{\text{in}}$ = measured free momentum eigenstate,

$|f\rangle_{\text{out}}$ = state $|f\rangle_{\text{in}}$ evolved back in time to interfere with $|i\rangle_{\text{in}}$.

- Asymptotic field operators:

$$\phi_{\text{in}}(x) = S\phi_{\text{out}}(x)S^\dagger, \quad (6.9)$$

so that

$${}_{\text{in}}\langle\alpha|\phi_{\text{in}}|\beta\rangle_{\text{in}} = {}_{\text{in}}\langle\alpha|S\phi_{\text{out}}S^\dagger|\beta\rangle_{\text{in}} = {}_{\text{out}}\langle\alpha|\phi_{\text{out}}|\beta\rangle_{\text{out}}, \quad \text{etc.} \quad (6.10)$$

- Poincaré invariance of matrix elements requires:

$$S = U(\Lambda, a) S U^\dagger(\Lambda, a), \quad (6.11)$$

Aim: perturbative expansion for S_{fi}

\Leftrightarrow derive relation between S , time evolution, and H

6.2 Perturbation Theory

Recapitulation of qm. time evolution pictures:

1. Schrödinger picture:

States carry time evolution, described by the *time evolution operator* U :

$$|\psi(t)\rangle = U(t, t_0) |\psi(t_0)\rangle, \quad (6.12)$$

$$i \frac{dU(t, t_0)}{dt} = H(t)U(t, t_0), \quad U(t_0, t_0) = 1. \quad (6.13)$$

Properties of U :

- unitarity:

$$U(t, t_0) = U^{-1}(t_0, t) = U^\dagger(t_0, t), \quad (6.14)$$

- group composition law:

$$U(t, t')U(t', t_0) = U(t, t_0). \quad (6.15)$$

Iterative solution:

$$\begin{aligned} U(t, t_0) &= 1 - i \int_{t_0}^t dt' H(t')U(t', t_0) \\ &= 1 - i \int_{t_0}^t dt' H(t') + (-i)^2 \int_{t_0}^t dt' \int_{t_0}^{t'} dt'' H(t')H(t'') + \dots \\ &= 1 - i \int_{t_0}^t dt' H(t') + \frac{(-i)^2}{2} \int_{t_0}^t dt' \left[\int_{t_0}^{t'} dt'' H(t')H(t'') + \underbrace{\int_{t'}^t dt'' H(t'')H(t')}_{\text{interchange integration variables } t' \leftrightarrow t''} \right] \\ &\quad + \dots \\ &= 1 - i \int_{t_0}^t dt' H(t') + \frac{1}{2}(-i)^2 \int_{t_0}^t dt' \int_{t_0}^{t'} dt'' T[H(t')H(t'')] + \dots \\ &\equiv T \exp \left[-i \int_{t_0}^t dt' H(t') \right]. \end{aligned}$$

2. Heisenberg picture:

States are time-independent and tied to the S picture at some time t_0 :

$$|\psi\rangle_{\text{H}} = |\psi(t_0)\rangle = U(t_0, t) |\psi(t)\rangle. \quad (6.16)$$

Operators are transformed in such a way that matrix elements remain the same:

$$O_{\text{H}}(t) = U^\dagger(t, t_0) O(t) U(t, t_0). \quad (6.17)$$

⇒ EOM for operators:

$$\begin{aligned} i\frac{dO_{\text{H}}(t)}{dt} &= i\left(\dot{U}^\dagger O(t)U + U^\dagger O(t)\dot{U}\right) + U^\dagger i\frac{\partial O(t)}{\partial t}U \\ &= [O_{\text{H}}(t), H_{\text{H}}(t)] + i\left(\frac{\partial O(t)}{\partial t}\right)_{\text{H}} \end{aligned} \quad (6.18)$$

$$\text{with } H_{\text{H}}(t) = U^\dagger(t, t_0) H(t) U(t, t_0).$$

3. Interaction picture:

- Hamilton operator split into a free and an interacting part,

$$H(t) = H_0(t) + H_{\text{int}}(t) \quad (\text{spectrum of } H_0(t) \text{ known}) \quad (6.19)$$

- Time evolution from H_0 removed from states (as in H picture):

$$|\psi(t)\rangle_{\text{I}} = U_0^\dagger(t, t_0) |\psi(t)\rangle, \quad O_{\text{I}}(t) = U_0^\dagger(t, t_0) O U_0(t, t_0) \quad (6.20)$$

with the time evolution operator U_0 of H_0 ,

$$i\frac{dU_0(t, t_0)}{dt} = H_0(t)U_0(t, t_0). \quad (6.21)$$

- EOM of the states:

$$i\frac{d}{dt} |\psi(t)\rangle_{\text{I}} = H_{\text{I}}(t) |\psi(t)\rangle_{\text{I}} \quad \text{with } H_{\text{I}}(t) = U_0^\dagger(t, t_0) H_{\text{int}}(t) U_0(t, t_0), \quad (6.22)$$

i.e. states evolve in time with the interaction Hamiltonian H_{I} .

- EOM of the operators:

$$i\frac{dO_{\text{I}}(t)}{dt} = [O_{\text{I}}(t), H_{0,\text{I}}(t)] + i\left(\frac{\partial O(t)}{\partial t}\right)_{\text{I}} \quad \text{with } H_{0,\text{I}}(t) = U_0^\dagger(t, t_0) H_0(t) U_0(t, t_0), \quad (6.23)$$

i.e. operators evolve in time with the free Hamiltonian $H_{0,\text{I}}$.

- Time evolution operator U_{I} in the I picture:

$$|\psi(t)\rangle_{\text{I}} = U_0^\dagger(t, t_0) U(t, t_0) |\psi\rangle_{\text{H}} \equiv U_{\text{I}}(t, t_0) |\psi\rangle_{\text{H}}, \quad (6.24)$$

$$O_{\text{I}}(t) = U_{\text{I}}(t, t_0) O_{\text{H}}(t) U_{\text{I}}^\dagger(t, t_0), \quad (6.25)$$

$$i\frac{dU_{\text{I}}(t, t_0)}{dt} = U_0^\dagger(t, t_0) H_{\text{int}}(t) U(t, t_0) = H_{\text{I}}(t) U_{\text{I}}(t, t_0). \quad (6.26)$$

Formal solution:

$$U_{\text{I}}(t, t_0) = T \exp \left[-i \int_{t_0}^t dt' H_{\text{I}}(t') \right]. \quad (6.27)$$

Application to scattering in QFT:

Comment:
 Here we assume $Z = 1$ and $\bar{m} = m$, which is sufficient for the lowest perturbative order and, thus, for this lecture.

- Initial states: take $t, t_0 < -T$ and subsequently $-T \rightarrow -\infty$.

◇ H picture and I picture are identical (no interaction yet).

◇ $|\psi(t)\rangle_{\text{I}} = |\psi(t_0)\rangle_{\text{H}} = |i\rangle_{\text{in}} = \text{const.}$

◇ $U_{\text{I}}(t, t_0) = \mathbf{1}, \quad \phi_{\text{in}}(x) = \phi(x).$

- Interaction period: $t_0 < -T < t < T$.

◇ $|\psi(t)\rangle_{\text{I}} = U_{\text{I}}(t, t_0) |i\rangle_{\text{in}} = U_{\text{I}}(t, -T) |i\rangle_{\text{in}}.$

◇ $\underbrace{\phi_{\text{in}}(x)}_{\text{I picture}} = U_{\text{I}}(t, -T) \underbrace{\phi(x)}_{\text{H picture}} U_{\text{I}}^\dagger(t, -T).$

- Final states: $t_0 < -T < T < t$.

◇ $|\psi(t)\rangle_{\text{I}} = U_{\text{I}}(t, t_0) |i\rangle_{\text{in}} = U_{\text{I}}(T, -T) |i\rangle_{\text{in}} \xrightarrow{T \rightarrow \infty} S |i\rangle_{\text{in}},$ where

$$S \equiv U_{\text{I}}(\infty, -\infty). \quad (6.28)$$

◇ $\phi_{\text{in}}(x) = U_{\text{I}}(t, -T) \phi(x) U_{\text{I}}^\dagger(t, -T) = U_{\text{I}}(T, -T) \phi_{\text{out}}(x) U_{\text{I}}^\dagger(T, -T)$
 $\xrightarrow{T \rightarrow \infty} S \phi_{\text{out}}(x) S^\dagger.$

◇ $|f(t)\rangle_{\text{I}} \equiv U_{\text{I}}(t, +T) |f\rangle_{\text{in}} = \text{test final state, where } |f\rangle_{\text{in}} \text{ typically is some measured free momentum eigenstate.}$

- S -matrix element:

$$\begin{aligned} {}_{\text{I}} \langle f(t) | \psi(t) \rangle_{\text{I}} &= {}_{\text{in}} \langle f | \underbrace{U_{\text{I}}^\dagger(t, T) U_{\text{I}}(t, -T)}_{= U_{\text{I}}(T, -T) \rightarrow S} |i\rangle_{\text{in}} \\ &\longrightarrow {}_{\text{in}} \langle f | S |i\rangle_{\text{in}} = {}_{\text{out}} \langle f |i\rangle_{\text{in}} = S_{fi}. \end{aligned} \quad (6.29)$$

Note: $|i\rangle_{\text{in}}$ and $|f\rangle_{\text{out}}$ are Heisenberg states corresponding to $t \rightarrow -\infty$.

- Operators in the I picture:

◇ Field operators (for all times t , see above) for free particle propagation:

$$\phi_{\text{in}}(x) = U_{\text{I}}(t) \phi(x) U_{\text{I}}^\dagger(t). \quad (6.30)$$

◇ Hamiltonian (needed for time evolution):

$$\begin{aligned}
 H_I(t) &= U_I(t)H_{\text{int}}(t)U_I^\dagger(t), & U_I(t) &\equiv U_I(t, -\infty) \\
 &= \int d^3x U_I(t) \mathcal{H}_{\text{int}}(\phi(x), \pi(x)) U_I^\dagger(t). \\
 &= \int d^3x \mathcal{H}_{\text{int}}(\phi_{\text{in}}(x), \pi_{\text{in}}(x)).
 \end{aligned} \tag{6.31}$$

Example: scalar field theory with interaction potential, $\mathcal{L}_{\text{int}}(\phi) = -V(\phi)$

$$H_I(t) = \int d^3x U_I(t) \mathcal{H}_{\text{int}}(\phi(x)) U_I^\dagger(t) = \int d^3x \mathcal{H}_{\text{int}}(\phi_{\text{in}}(x)) \tag{6.32}$$

with

$$\mathcal{H}_{\text{int}}(\phi_{\text{in}}) = V(\phi_{\text{in}}) = -\mathcal{L}_{\text{int}}(\phi_{\text{in}}). \tag{6.33}$$

|| Comment:

This transition is more complicated if the interaction $V(\phi)$ involves derivatives, i.e. if \mathcal{H}_{int} involves canonical momenta. Then, in general, $\mathcal{H}_{\text{int}}(\phi_I) \neq -\mathcal{L}_{\text{int}}(\phi_I)$, as e.g. in scalar QED.

- Perturbative expansion of S -matrix: ($|i\rangle \equiv |i\rangle_{\text{in}}$, $|f\rangle \equiv |f\rangle_{\text{in}}$)

$$S_{fi} = \langle f| T \exp \left[-i \int d^4x \mathcal{H}_{\text{int}}(\phi_{\text{in}}(x)) \right] |i\rangle \tag{6.34}$$

$$\begin{aligned}
 &= \langle f|i\rangle - i \int d^4x \langle f| \mathcal{H}_{\text{int}}(\phi_{\text{in}}(x)) |i\rangle \\
 &\quad + \sum_{n=2}^{\infty} \frac{(-i)^n}{n!} \left(\prod_{j=1}^n \int d^4x_j \right) \langle f| T [\mathcal{H}_{\text{int}}(\phi_{\text{in}}(x_1)) \dots \mathcal{H}_{\text{int}}(\phi_{\text{in}}(x_n))] |i\rangle.
 \end{aligned} \tag{6.35}$$

6.3 Feynman diagrams

6.3.1 Wick's theorem

Task: Computation of S -matrix elements, i.e. of

$$\langle f | T[\mathcal{H}_{\text{int}}(x_1) \dots \mathcal{H}_{\text{int}}(x_n)] | i \rangle, \quad (6.36)$$

where $\mathcal{H}_{\text{int}}(x) = : \phi_{\text{in}}(x) \phi_{\text{in}}^\dagger(x) \dots :$ with free “in”-fields ϕ_{in} , etc.

\Rightarrow Translate time-ordered products into normal-ordered products, so that $\langle f | a^\dagger(p) \dots a(k) | i \rangle$ can be evaluated explicitly.

Example used for illustration: one real scalar field ϕ with

$$\mathcal{H}_{\text{int}}(x) = -\frac{g}{3!} : \phi^3(x) : . \quad (6.37)$$

Note: Subscript “in” suppressed in the following.

Definition: *Contraction* of free, bosonic real field operators:

$$: \dots \overbrace{\phi_i \dots \phi_j} \dots : = \langle 0 | T[\phi_i \phi_j] | 0 \rangle \cdot : \dots \phi_{i-1} \phi_{i+1} \dots \phi_{j-1} \dots \phi_{j+1} \dots : \quad (6.38)$$

Example:

$$\overbrace{\phi(x) \phi(y)} = \underbrace{\langle 0 | T[\phi(x) \phi(y)] | 0 \rangle}_{= \text{complex number}} = iD_{\text{F}}(x, y), \quad \text{Feynman propagator} \quad (6.39)$$

Identity for time-ordered product of two field operators (see Exercise 6.2):

$$\begin{aligned} T[\phi(x) \phi(y)] &= : \phi(x) \phi(y) : + \langle 0 | T[\phi(x) \phi(y)] | 0 \rangle \\ &= : \phi(x) \phi(y) : + \overbrace{\phi(x) \phi(y)}. \end{aligned} \quad (6.40)$$

General case of an arbitrary number of free, bosonic, real field operators ruled by

Wick's theorem: (Discussion in Exercise 7.1)

$$\begin{aligned} T[\phi_1 \dots \phi_n] &= : \phi_1 \dots \phi_n : + \sum_{\text{pairs } ij} : \phi_1 \dots \overbrace{\phi_i \dots \phi_j} \dots \phi_n : \\ &+ \sum_{\text{double pairs } ij,kl} : \phi_1 \dots \phi_i \dots \overbrace{\phi_k \dots \phi_j} \dots \overbrace{\phi_l} \dots \phi_n : + \dots \end{aligned} \quad (6.41)$$

Extension: If some of the fields in the argument of the T -product in (6.41) are within the same normal-ordered product, they will not be contracted.

Examples:

- four fields:

$$\begin{aligned}
 T[\phi_1 \cdots \phi_4] = & : \phi_1 \cdots \phi_4 : + : \phi_1 \phi_2 : \overbrace{\phi_3 \phi_4} + : \phi_1 \phi_3 : \overbrace{\phi_2 \phi_4} + : \phi_1 \phi_4 : \overbrace{\phi_2 \phi_3} \\
 & + : \phi_2 \phi_3 : \overbrace{\phi_1 \phi_4} + : \phi_2 \phi_4 : \overbrace{\phi_1 \phi_3} + : \phi_3 \phi_4 : \overbrace{\phi_1 \phi_2} \\
 & + \overbrace{\phi_1 \phi_2 \phi_3 \phi_4} + \overbrace{\phi_1 \phi_3 \phi_2 \phi_4} + \overbrace{\phi_1 \phi_4 \phi_2 \phi_3},
 \end{aligned} \tag{6.42}$$

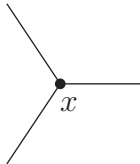
- pair of normal-ordered products:

$$\begin{aligned}
 T[: \phi_1 \phi_2 : : \phi_3 \phi_4 :] = & : \phi_1 \cdots \phi_4 : + : \phi_1 \phi_3 : \overbrace{\phi_2 \phi_4} + : \phi_1 \phi_4 : \overbrace{\phi_2 \phi_3} \\
 & + : \phi_2 \phi_3 : \overbrace{\phi_1 \phi_4} + : \phi_2 \phi_4 : \overbrace{\phi_1 \phi_3} + \overbrace{\phi_1 \phi_3 \phi_2 \phi_4} + \overbrace{\phi_1 \phi_4 \phi_2 \phi_3}.
 \end{aligned} \tag{6.43}$$

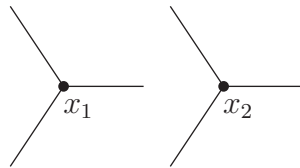
6.3.2 Feynman rules for the S-operator

Application of Wick theorem expands S operator in terms of propagators and normal-ordered fields:

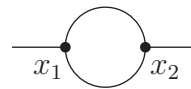
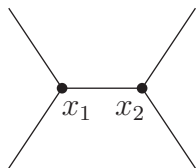
$$\begin{aligned}
 S &= U_I(\infty, -\infty) = T \exp \left[\int d^4x \frac{ig}{3!} : \phi^3(x) : \right] \\
 &= \mathbf{1} + \frac{ig}{3!} \int d^4x : \phi^3(x) :
 \end{aligned}$$



$$+ \frac{1}{2} \left(\frac{ig}{3!} \right)^2 \int d^4x_1 d^4x_2 \left[: \phi^3(x_1) \phi^3(x_2) : \right.$$



$$+ 3^2 : \phi^2(x_1) \phi^2(x_2) : \overbrace{\phi(x_1) \phi(x_2)} + 2(3)^2 : \phi(x_1) \phi(x_2) : \left(\overbrace{\phi(x_1) \phi(x_2)} \right)^2$$



$$+ 3! \left(\overbrace{\phi(x_1)\phi(x_2)} \right)^3 \Big] + \dots \tag{6.44}$$



$$\tag{6.45}$$

Feynman rules for graphical representation of the terms $\propto g^n$:

1. Draw all possible diagrams with n 3-point vertices, connected in all possible ways by lines (including disconnected diagrams).
2. Translate graphs into analytical expressions as follows:
 - External lines for non-contracted fields:

$$\phi(x) = \text{---}\bullet_x \tag{6.46}$$

- Internal lines for contracted fields (=propagators):

$$\overbrace{\phi(x_1)\phi(x_2)} = \bullet_{x_1}\text{---}\bullet_{x_2} \tag{6.47}$$

- Vertices for interaction terms:



$$\frac{ig}{3!} = \tag{6.48}$$

3. Include a combinatorial factor (*symmetry factor*) for each diagram (more detail as explained below).
4. Integrate the sum of all terms according to

$$\frac{1}{n!} \int d^4x_1 \dots d^4x_n : \dots : \tag{6.49}$$

6.3.3 Feynman rules for S-matrix elements

Final task: Evaluate $\langle f|S|i\rangle$ upon sandwiching expansion of S -operator between states

$$|i\rangle = |\vec{k}_1, \dots, \vec{k}_m\rangle, \quad |f\rangle = |\vec{p}_1, \dots, \vec{p}_n\rangle. \tag{6.50}$$

Definitions:

- *T-matrix*

$$\langle f|S|i\rangle = \underbrace{\langle f|i\rangle}_{= 0 \text{ for } |i\rangle \neq |f\rangle, \text{ unscattered part}} + \underbrace{\langle f|S - \mathbf{1}|i\rangle}_{\equiv \langle f|T|i\rangle, T\text{-matrix, only scattered part}}. \quad (6.51)$$

\hookrightarrow Only the T -matrix contributes to a non-trivial scattering with $|i\rangle \neq |f\rangle$.

- *Transition matrix element (transition amplitude)*

$$\langle f|T|i\rangle = i(2\pi)^4 \underbrace{\delta\left(\sum_i k_i - \sum_j p_j\right)}_{\text{expresses momentum conservation, due to translational invariance}} \underbrace{\mathcal{M}_{fi}}_{\text{transition matrix element}}. \quad (6.52)$$

Example: $2 \rightarrow 2$ scattering

$$|i\rangle = |\vec{k}_1, \vec{k}_2\rangle, \quad |f\rangle = |\vec{p}_1, \vec{p}_2\rangle. \quad (6.53)$$

Use the expansion of the S matrix (6.44).

$\hookrightarrow \langle f|T|i\rangle$ involves expectation values of normal-ordered operator products:

$$\langle \vec{p}_1, \vec{p}_2 | : \phi^3(x) : | \vec{k}_1, \vec{k}_2 \rangle, \quad \langle \vec{p}_1, \vec{p}_2 | : \phi^n(x_1) \phi^n(x_2) : | \vec{k}_1, \vec{k}_2 \rangle, \quad n = 0, 1, 2, 3. \quad (6.54)$$

Recall: $\phi = \phi_{\text{in}} =$ free field.

\hookrightarrow Use plane-wave decomposition with creation/annihilation operators $a^\dagger(\vec{p})/a(\vec{p})$:

$$\phi(x) = \int d\vec{p} [a(\vec{p}) e^{-ipx} + a^\dagger(\vec{p}) e^{ipx}]. \quad (6.55)$$

\Rightarrow Only operator combinations with two a 's and two a^\dagger 's contribute in $\langle \vec{p}_1, \vec{p}_2 | \dots | \vec{k}_1, \vec{k}_2 \rangle$, i.e.

$$\langle \vec{p}_1, \vec{p}_2 | : \phi^2(x_1) \phi^2(x_2) : | \vec{k}_1, \vec{k}_2 \rangle. \quad (6.56)$$

Typical manipulation:

$$\begin{aligned} \dots \phi(x_i) \phi(x_j) : | \vec{k}_1, \vec{k}_2 \rangle &= \int d\vec{q}_1 d\vec{q}_2 e^{-iq_1 x_i} e^{-iq_1 x_j} a(\vec{q}_1) a(\vec{q}_2) \underbrace{| \vec{k}_1, \vec{k}_2 \rangle}_{= a^\dagger(\vec{k}_1) a^\dagger(\vec{k}_2) | 0 \rangle} + \dots \\ &= \int d^3 q_1 d^3 q_2 e^{-iq_1 x_i} e^{-iq_1 x_j} \left[\delta(\vec{q}_2 - \vec{k}_1) \delta(\vec{q}_1 - \vec{k}_2) + \delta(\vec{q}_2 - \vec{k}_2) \delta(\vec{q}_1 - \vec{k}_1) \right] | 0 \rangle + \dots \\ &= (e^{-ik_1 x_i} e^{-ik_2 x_j} + e^{-ik_1 x_j} e^{-ik_2 x_i}) | 0 \rangle + \dots \end{aligned} \quad (6.57)$$

General case:

- Define contractions of fields with external states:

$$:\dots\overbrace{\phi(x)\dots}^{\quad}:\dots\overbrace{\vec{k}\dots}^{\quad}\rangle = e^{-ikx}, \quad (6.58)$$

$$\langle\dots\overbrace{\vec{p}\dots}^{\quad}:\dots\overbrace{\phi(x)\dots}^{\quad}:\rangle = e^{ipx}. \quad (6.59)$$

- Perform all possible contractions of fields in normal-ordered products with external states.

Application to example $\langle\vec{p}_1, \vec{p}_2 | : \phi^2(x_1)\phi^2(x_2) : | \vec{k}_1, \vec{k}_2 \rangle$:

- Three types of contractions:

$$\langle\overbrace{\vec{p}_1, \vec{p}_2}^{\quad} : \overbrace{\phi^2(x_1)\phi^2(x_2)}^{\quad} : \overbrace{|\vec{k}_1, \vec{k}_2\rangle}^{\quad}\rangle = 2^2 e^{i(p_1+p_2)x_1} e^{-i(k_1+k_2)x_2}, \quad (6.60)$$

$$\langle\overbrace{\vec{p}_1, \vec{p}_2}^{\quad} : \overbrace{\phi^2(x_1)\phi^2(x_2)}^{\quad} : \overbrace{|\vec{k}_1, \vec{k}_2\rangle}^{\quad}\rangle = 2^2 e^{-i(k_1-p_1)x_1} e^{-i(k_2-p_2)x_2}, \quad (6.61)$$

$$\langle\overbrace{\vec{p}_1, \vec{p}_2}^{\quad} : \overbrace{\phi^2(x_1)\phi^2(x_2)}^{\quad} : \overbrace{|\vec{k}_1, \vec{k}_2\rangle}^{\quad}\rangle = 2^2 e^{-i(k_1-p_2)x_1} e^{-i(k_2-p_1)x_2}, \quad (6.62)$$

plus the identical contributions with $x_1 \leftrightarrow x_2$. \Rightarrow Factor of 2.

- Apply remaining factors and integrals for T -matrix element:

$$\frac{1}{2} 3^2 \frac{(ig)^2}{(3!)^2} \int d^4x_1 d^4x_2 \langle\vec{p}_1, \vec{p}_2 | : \phi^2(x_1)\phi^2(x_2) : | \vec{k}_1, \vec{k}_2 \rangle \overbrace{\phi(x_1)\phi(x_2)}^{\quad} \rangle \quad (6.63)$$

and use the momentum-space representation of propagator,

$$\overbrace{\phi(x_1)\phi(x_2)}^{\quad} = \langle 0 | T [\phi(x_1)\phi(x_2)] | 0 \rangle = \int \frac{d^4q}{(2\pi)^4} \frac{i}{q^2 - m^2 + i\epsilon} e^{-iq(x_1-x_2)}. \quad (6.64)$$

Example: Explicit evaluation of contribution (6.62):

$$\begin{aligned} & \frac{1}{2} 3^2 \frac{(ig)^2}{(3!)^2} \int d^4x_1 d^4x_2 \int \frac{d^4q}{(2\pi)^4} \frac{i}{q^2 - m^2 + i\epsilon} e^{-iq(x_1-x_2)} 2^3 e^{-i(k_1-p_2)x_1} e^{-i(k_2-p_1)x_2} \\ &= (ig)^2 \int \frac{d^4q}{(2\pi)^4} (2\pi)^4 \delta(q + k_1 - p_2) \frac{i}{q^2 - m^2 + i\epsilon} (2\pi)^4 \delta(q + p_1 - k_2) \\ &= (2\pi)^4 \delta(k_1 + k_2 - p_1 - p_2) (ig)^2 \frac{i}{(k_1 - p_2)^2 - m^2 + i\epsilon}, \end{aligned} \quad (6.65)$$

\hookrightarrow δ -function for momentum conservation appears explicitly [cf. (6.52)]; all combinatorial prefactors have canceled.

- Sum of the three different contractions (6.60)–(6.62):

$$\begin{aligned}
 i\mathcal{M}_{fi} &= (ig)^2 \left[\frac{i}{(k_1 + k_2)^2 - m^2 + i\epsilon} + \frac{i}{(k_1 - p_1)^2 - m^2 + i\epsilon} + \frac{i}{(k_1 - p_2)^2 - m^2 + i\epsilon} \right] \\
 &= \begin{array}{c} \begin{array}{ccc} k_1 & & p_1 \\ & \diagdown & / \\ & \bullet & \bullet \\ & / & \diagdown \\ k_2 & & p_2 \end{array} & + & \begin{array}{ccc} k_1 & & p_1 \\ & \diagdown & / \\ & \bullet & \bullet \\ & | & \\ & \bullet & \\ & / & \diagdown \\ k_2 & & p_2 \end{array} & + & \begin{array}{ccc} k_1 & & p_1 \\ & \diagdown & / \\ & \bullet & \bullet \\ & / & \diagdown \\ k_2 & & p_2 \end{array} \end{array} \\
 &= (ig)^2 \left[\frac{i}{s - m^2 + i\epsilon} + \frac{i}{t - m^2 + i\epsilon} + \frac{i}{u - m^2 + i\epsilon} \right], \tag{6.66}
 \end{aligned}$$

with the three Mandelstam variables (see Exercise 2.3):

$$s = (k_1 + k_2)^2, \quad t = (k_1 - p_1)^2, \quad u = (k_1 - p_2)^2. \tag{6.67}$$

Note: One diagram of the expansion (6.44) of the S -operator produces three diagrams in the expansion of the transition matrix element.

Generalization of the $2 \rightarrow 2$ example to arbitrary processes leads to

Feynman rules transition matrix elements in momentum space

for contributions proportional to g^N term in $i\mathcal{M}_{fi}$ for an $n \rightarrow m$ scattering process:

1. Draw all possible diagrams with N 3-point vertices and n incoming and m outgoing external legs.
2. Impose momentum conservation at each vertex.
3. Insert the following expressions:

- external lines:

$$\text{---} \bullet = 1 \tag{6.68}$$

- internal lines:

$$\bullet \text{---} \bullet = \frac{i}{q^2 - m^2 + i\epsilon} \tag{6.69}$$

- vertices:

$$\begin{array}{ccc} & \diagdown & / \\ & \bullet & \\ & / & \diagdown \end{array} = ig \tag{6.70}$$

4. Apply a symmetry factor $1/S_G$ for each diagram (see below).
5. Integrate over (loop) momenta q_i not fixed by momentum conservation according to

$$\int \frac{d^4 q_i}{(2\pi)^4}. \quad (6.71)$$

Note: In our $2 \rightarrow 2$ example $S_G = 1$, and all momenta were fixed by external momenta.

Comments to the general case:

- *Loop momenta* (not fixed by external states):

Systematic counting:

- 1 momentum integral per propagator.
- 1 space-time integral per vertex (yields momentum conservation at vertex).

Above example: $\int d^4 x_1 e^{-i(q+k_1-p_2)x_1} = (2\pi)^4 \delta(q+k_1-p_2)$.

- δ -function for overall momentum conservation split off from \mathcal{M}_{fi} .

\Rightarrow remaining # momentum integrals is given by

$$L = \# \text{propagators} - \# \text{vertices} + 1 = \text{number of loops in a diagram}. \quad (6.72)$$

\Rightarrow Perturbation series for \mathcal{M}_{fi} is an expansion in # loops:

- $L = 0$, leading order, *Born approximation*,
- $L = 1$, next-to-leading order, *one-loop approximation*,
- ...

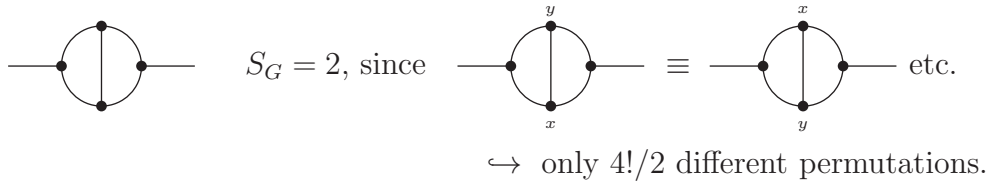
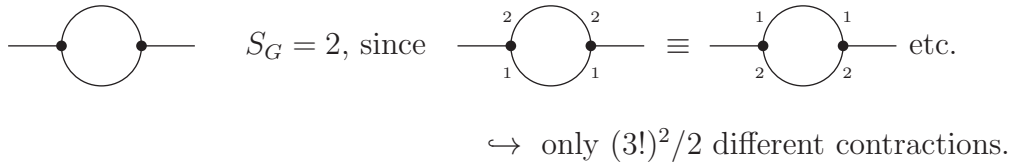
- *Symmetry factor* S_G :

$S_G \neq 1$ results from two sources:

- Incomplete cancellation of factor $1/3!$ in $\mathcal{H}_{\text{int}}(x) = -\frac{g}{3!} : \phi^3(x) :$,
because some contractions to propagators are diagrammatically equivalent.
- Incomplete cancellation of factor $1/n!$ in n th term of $S = T \exp\{\dots\}$,
because some permutations of vertices x_i are diagrammatically equivalent.

$S_G = \#$ permutations of internal lines or vertices that leaves the diagram unchanged
= the order of the symmetry group of graph G .

Examples:

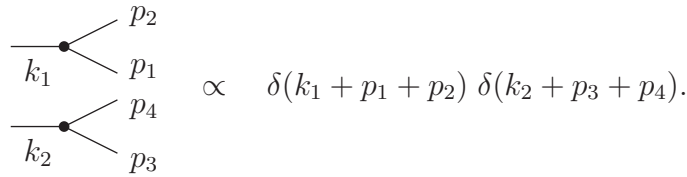


- *Disconnected diagrams:*

Momentum conservation at each vertex

\leftrightarrow “overall momentum conservation” in each connected part of a diagram.

Example:

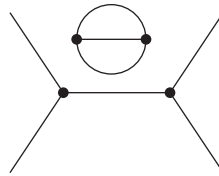


Contributions only for *exceptional momenta*, corresponding to different scattering proceeding in parallel.

\Rightarrow Contributions irrelevant for single scattering reaction.

- *Vacuum diagrams:*

= (sub)diagrams with no external legs, e.g.:



– Vacuum graphs always factor from remaining graphs of the diagram.

\leftrightarrow They modify each S -matrix element S_{fi} by the same factor.

– Vacuum correction factor calculable as

$$\langle 0|S|0\rangle = \langle 0|U_I(\infty, -\infty)|0\rangle = \text{phase factor}, \tag{6.73}$$

contradicting the initial assumption (6.7) $|0\rangle = S|0\rangle$.

⇒ Redefine

$$S = \frac{U_I(\infty, -\infty)}{\langle 0|U_I(\infty, -\infty)|0\rangle},$$

so that $\langle 0|S|0\rangle = 1$ and all vacuum graphs cancel in observables (i.e. they can be ignored in practice).

6.4 Cross sections and decay widths

Cross section:

Definition:

$$\underbrace{dN_s}_{\substack{\# \text{ scattering events / time} \\ \text{with particle } f_l \text{ carrying} \\ \text{momentum } \vec{p}_{f_l}}} = \underbrace{d\sigma}_{\substack{\text{differential} \\ \text{cross section}}} \times \underbrace{F}_{\substack{\text{incoming particle flux} \\ = \frac{\# \text{interactions}}{\text{area} \times \text{time}}}},$$

where

- $F = \frac{N_1 N_2 v_{\text{rel}}}{V}$ with $N_{1,2} = \#$ incoming particles type 1, 2 and v_{rel} = relative velocity between incoming particles.

- $dN_s = \underbrace{\frac{W_{fi}}{T}}_{\text{transition rate}} \prod_{l=1}^n \left(V \frac{d^3 p_{f_l}}{(2\pi)^3} \right) N_1 N_2,$

with $V \frac{d^3 p_{f_l}}{(2\pi)^3} = \#$ states of stationary waves in a box with volume V .

- Transition probability W_{fi} for $|i\rangle \rightarrow |f\rangle$:

$$W_{fi} = \frac{|\langle f|T|i\rangle|^2}{\langle f|f\rangle \langle i|i\rangle} \tag{6.74}$$

which is not well defined as $|\langle f|T|i\rangle|^2 \propto [\delta(p_i - p_f)]^2$ and $\langle f|f\rangle, \langle i|i\rangle \rightarrow \infty$.

Solution:

Box with finite extension in space–time, volume = $V \cdot T$:

$$\begin{aligned} (2\pi)^8 [\delta(p_i - p_f)]^2 &\longrightarrow V \cdot T (2\pi)^4 \delta(p_i - p_f), \\ \langle \vec{p}|\vec{p}'\rangle = (2\pi)^3 2p^0 \delta(\vec{p} - \vec{p}') &\xrightarrow{\vec{p}' \rightarrow \vec{p}} 2p^0 V. \end{aligned} \tag{6.75}$$

$$\begin{aligned} \Rightarrow W_{fi} &= V \cdot T (2\pi)^4 \delta(p_i - p_f) \frac{1}{(2p_{i_1}^0 V)(2p_{i_2}^0 V)} \left(\prod_{l=1}^n \frac{1}{(2p_{f_l}^0 V)} \right) \underbrace{|\mathcal{M}_{fi}|^2}_{\text{transition matrix element}}, \\ d\sigma &= \underbrace{\frac{1}{4p_{i_2}^0 p_{i_2}^0 v_{\text{rel}}}}_{\text{flux factor}} |\mathcal{M}_{fi}|^2 \underbrace{\left(\prod_{l=1}^n \frac{d^3 p_{f_l}}{2p_{f_l}^0 (2\pi)^3} \right)}_{= d\Phi_f, \text{ invariant phase space volume}} (2\pi)^4 \delta(p_i - p_f). \end{aligned} \quad (6.76)$$

Using the Lorentz-invariant form of F ,

$$p_{i_1}^0 p_{i_2}^0 v_{\text{rel}} = \sqrt{(p_{i_1} p_{i_2})^2 - (m_{i_1} m_{i_2})^2}, \quad \text{with } p_{i_1}^2 = m_{i_1}^2, p_{i_2}^2 = m_{i_2}^2,$$

$d\sigma$ takes the final form:

$$d\sigma = \frac{1}{4\sqrt{(p_{i_1} p_{i_2})^2 - (m_{i_1} m_{i_2})^2}} |\mathcal{M}_{fi}|^2 d\Phi_f. \quad (6.77)$$

\Rightarrow Total cross section:

$$\sigma_{\text{tot}} = \int d\sigma = \frac{1}{4\sqrt{(p_{i_1} p_{i_2})^2 - (m_{i_1} m_{i_2})^2}} \int d\Phi_f |\mathcal{M}_{fi}|^2. \quad (6.78)$$

Comments:

- $d\sigma$ is Lorentz invariant.
- For polarizable particles (\neq scalars):
 - initial state: take specific polarization or average over incoming spin states,
 - final state: analyze specific polarization or sum over outgoing spin states.
- Identical particles in final state: exclude identical configurations in $\int d\Phi_f$.
 \hookrightarrow Factor $1/(n_x!)$ for n_x identical particles of type X in full integral.
- Differential cross sections:
 Leave one or more kinematical variables in $\int d\Phi_f$ open.
 Example: Distribution in scattering angle θ for the particle f_1 :

$$\frac{d\sigma}{d\theta} = \int d\sigma \delta(\theta - \theta_{f_1}).$$

Decay width:

Particle decay: $X \rightarrow f_1 + \cdots + f_n$

\Leftrightarrow Treatment analogous to scattering!

Results:

- Partial decay width:

$$\Gamma_{X \rightarrow f} = \frac{1}{2m_X} \int d\Phi_f |\mathcal{M}_{fX}|^2. \quad (6.79)$$

- Total decay width:

$$\Gamma_{\text{tot}} = \underbrace{\sum_f}_{\substack{\text{sum over all} \\ \text{decay channels}}} \Gamma_{X \rightarrow f}, \quad (6.80)$$

$$\Rightarrow \text{Particle lifetime: } \tau_X = \frac{\hbar}{\Gamma_{\text{tot}}}.$$

Note: Treatment of polarizations, identical particles, and differential distributions analogous to cross section.

Example: $\phi\phi$ scattering in ϕ^3 theory in lowest order

Process:

$$\phi(k_1) \phi(k_2) \rightarrow \phi(p_1) \phi(p_2), \quad \text{momentum conservation: } k_1 + k_2 = p_1 + p_2. \quad (6.81)$$

- Momenta in centre-of-mass frame:

$$k_{1,2}^\mu = (E, 0, 0, \pm\beta E), \quad k_{1,2}^2 = m^2, \quad (6.82)$$

$$\text{with } E = \text{beam energy}, \quad \beta = \sqrt{1 - \frac{m^2}{E^2}} = \text{velocity},$$

$$p_{1,2}^\mu = (E, \pm E\beta \sin\theta \cos\varphi, \pm E\beta \sin\theta \sin\varphi, \pm E\beta \cos\theta), \quad p_{1,2}^2 = m^2, \quad (6.83)$$

$$\text{with } \theta = \text{scattering angle} = \text{angle between } \vec{p}_1 \text{ and } \vec{k}_1.$$

- Mandelstam variables:

$$s = (k_1 + k_2)^2 = 4E^2, \quad (6.84)$$

$$t = (k_1 - p_1)^2 = p_1^2 + k_1^2 - 2p_1 \cdot k_1 = -2\beta^2 E^2 (1 - \cos\theta) = -4\beta^2 E^2 \sin^2 \frac{\theta}{2}, \quad (6.85)$$

$$u = (k_1 - p_2)^2 = \cdots = -4\beta^2 E^2 \cos^2 \frac{\theta}{2}, \quad (6.86)$$

$$s + t + u = 4m^2. \quad (6.87)$$

- Born diagrams:

$$i\mathcal{M}_{fi} = \begin{array}{c} \begin{array}{ccc} k_1 & & p_1 \\ & \diagdown & / \\ & \bullet & \\ & / & \diagdown \\ k_2 & & p_2 \end{array} & + & \begin{array}{ccc} k_1 & p_1 \\ & \bullet \\ & | \\ & \bullet \\ k_2 & p_2 \end{array} & + & \begin{array}{ccc} k_1 & & p_1 \\ & \diagdown & / \\ & \bullet & \\ & / & \diagdown \\ k_2 & & p_2 \end{array} \\ s\text{-channel} & & t\text{-channel} & & u\text{-channel} \end{array} \quad (6.88)$$

$$= (ig)^2 \left[\frac{i}{s-m^2} + \frac{i}{t-m^2} + \frac{i}{u-m^2} \right] \quad (6.89)$$

= dependent on E and θ , but not on φ (rotational invariance wrt beam axis!).

- Cross section:

$$- \text{flux} = \frac{1}{4\sqrt{(k_1 k_2)^2 - m^4}} = \frac{1}{4\sqrt{E^4(1+\beta^2)^2 - m^4}} = \frac{1}{8E^2\beta},$$

- phase space: (see Exercise 5.2)

$$\int d\Phi_2 = \frac{1}{(2\pi)^2} \frac{\sqrt{\lambda(s, m^2, m^2)}}{8s} \int d\Omega_1 = \frac{\beta}{32\pi^2} \int d\varphi \int d\cos\theta. \quad (6.90)$$

$$\begin{aligned} \Rightarrow \sigma &= \text{flux} \int d\Phi_2 |\mathcal{M}_{fi}|^2 \\ &= \frac{g^4}{256\pi^2 E^2} \underbrace{\int d\varphi}_{\rightarrow 2\pi} \underbrace{\int d\cos\theta \left[\frac{1}{s-m^2} + \frac{1}{t-m^2} + \frac{1}{u-m^2} \right]^2}_{\text{expressible in terms of logarithms}}. \end{aligned} \quad (6.91)$$

Part II

Quantization of fermion fields

Chapter 7

Representations of the Lorentz group

7.1 Lie groups and algebras

↔ Continuous groups (e.g. Lorentz / Poincaré groups, many internal symmetry groups)

7.1.1 Definitions

A *Lie group* is a group whose elements g are parametrized by a set of continuous parameters ω_a , $a = 1, \dots, n$: $g(\omega) = g(\omega_1, \dots, \omega_n)$.

- Group-multiplication law:

$$g(\omega)g(\omega') = g(\omega'') \quad \text{with} \quad \omega''_a = \underbrace{f_a(\omega_b, \omega'_c)}_{\text{differentiable functions of } \omega_b, \omega'_c}, \quad (7.1)$$

- n = dimension of the Lie group,
- identity e corresponds to $\omega_a = 0$ by convention:

$$g(0) = e. \quad (7.2)$$

- A Lie group is called *compact* if the set of all ω_a is compact.
- A Lie group is called *connected* if every element $g(\omega)$ is connected to the identity e by a continuous path in the set of the parameters ω .

Example: Lorentz group L

- L = non-compact (space of boosts is non-compact).
- $L \neq$ connected (disconnected parts characterized by $\det \Lambda = \pm 1$ and $\Lambda^0_0 \gtrless 0$);
 L_+^\uparrow = connected.

Comment:

Examples:

$GL(N, \mathbb{C})$: *general linear group*

The $2N^2$ -dimensional group of invertible complex matrices A ($\det A \neq 0$).

$SL(N, \mathbb{C})$: *special linear group*

The $(2N^2 - 2)$ -dimensional group of complex matrices A with $\det A = 1$.

$O(N)$: *orthogonal group*

The $N(N - 1)/2$ -dimensional group of real orthogonal matrices M ($M^T M = \mathbf{1}$, so that $\det M = \pm 1$).

$SO(N)$: *special orthogonal group*

The $N(N - 1)/2$ -dimensional group of orthogonal matrices M with $\det M = 1$.

$U(N)$: *unitary group*

The N^2 -dimensional group of unitary matrices U , $U^\dagger U = 1$.

$SU(N)$: *special unitary group*

The $(N^2 - 1)$ -dimensional group of unitary matrices U with $\det(U) = 1$.

Definition:

A *representation* D of a group G on a vector space V is a mapping of all $g \in G$ to linear transformations $D(g)$ on V that is compatible with group multiplication:

$$f \cdot g = h \quad \Rightarrow \quad D(f)D(g) = D(f \cdot g) = D(h). \quad (7.3)$$

- $V =$ *representation space*.
- $\dim V =$ dimension of the representation
(if $n = \dim V < \infty$, $D(g)$ are $n \times n$ -matrices).
- Elements $v \in V$ are called *multiplets* (at least in physics).
- Two representations D and D' are called *equivalent* ($D \sim D'$) if there is an invertible transformation S so that

$$D'(g) = SD(g)S^{-1}, \quad \forall g \in G.$$

- A representation is called *unitary* if the matrices $D(g)$ are unitary for all g .

7.1.2 Lie algebras

Note: Neighbourhood of identity carries almost full information about Lie group G .

Definition:

Lie algebra $\mathfrak{g} \equiv$ set of infinitesimal deviations from identity e
 \hookrightarrow vector space with product structure.

Properties of \mathfrak{g} in a specific representation $D(\mathfrak{g})$:

- Infinitesimal group elements $g = g(\delta\omega)$ in representation $D(G)$:

$$D(g) \equiv D(\delta\omega) = \mathbf{1} + \underbrace{\delta\omega_a \frac{\partial D(\omega)}{\partial \omega_a}}_{\equiv -iT_D^a} \Big|_{\omega=0} + \mathcal{O}(\delta\omega^2) = \mathbf{1} - iT_D^a \delta\omega_a + \mathcal{O}(\delta\omega^2). \quad (7.4)$$

$\{T_D^a\} =$ *generators* of the Lie group in D representation
 $=$ basis for representation $D(\mathfrak{g})$ of \mathfrak{g} .

- Finite transformations (connected to the unit element) via exponentiation:

$$D(\omega) = \lim_{n \rightarrow \infty} \left(1 - i \frac{\omega_a}{n} T^a\right)^n = \exp(-i\omega_a T^a), \quad (7.5)$$

- Composition law of G implies product in \mathfrak{g} :

Ansatz for composition functions of Eq. (7.1):

$$f_a(\omega_b, \omega'_c) = \omega_a + \omega'_a + \frac{1}{2} f^{abc} \omega_b \omega'_c + \dots, \quad (7.6)$$

i.e. $f_a(\omega_b, 0) = f_a(0, \omega_b) = \omega_a$.

\hookrightarrow Insertion into composition law:

$$\begin{aligned} D(\omega)D(\omega') &= \exp(-i\omega_a T^a) \exp(-i\omega'_b T^b) \\ &= \mathbf{1} - i\omega_a T^a - i\omega'_b T^b - \frac{1}{2}(\omega_a T^a)^2 - \frac{1}{2}(\omega'_b T^b)^2 - \omega_a \omega'_b T^a T^b + \dots \\ &\stackrel{!}{=} \exp(-i f_a(\omega, \omega') T^a) \\ &= \mathbf{1} - i \left(\omega_a + \omega'_a + \frac{1}{2} f^{abc} \omega_b \omega'_c \right) T^a - \frac{1}{2} (\omega_a + \omega'_a) (\omega_b + \omega'_b) T^a T^b + \dots \\ \text{i.e. } -\omega_a \omega'_b T^a T^b &= -\frac{i}{2} f^{abc} \omega_b \omega'_c T^a - \frac{1}{2} \omega_a \omega'_b T^a T^b - \frac{1}{2} \omega'_a \omega_b T^a T^b \end{aligned}$$

\Rightarrow Basic Lie algebra relation:

$$T^b T^c - T^c T^b \equiv [T^b, T^c] = i \underbrace{f^{abc}}_{\text{structure constants of } \mathfrak{g}} T^a, \quad \text{where } f^{abc} = -f^{acb}. \quad (7.7)$$

structure constants of $\mathfrak{g} =$ independent of representation !

- *Jacobi identity* of commutators,

$$[T^a, [T^b, T^c]] + [T^c, [T^a, T^b]] + [T^b, [T^c, T^a]] = 0, \quad (7.8)$$

implies

$$f^{abk} f^{kcd} + f^{ack} f^{kdb} + f^{adk} f^{kbc} = 0. \quad (7.9)$$

- *Adjoint representation*

$$(T_{\text{adj}}^a)_{bc} \equiv -i f^{bca} \quad (7.10)$$

exists for each Lie algebra.

Commutator relation (7.7) satisfied due to the Jacobi identity (7.9).

Important special case: algebra of a compact Lie group

- Matrix $\text{tr}(T^a T^b)$ is positive definite.

\Leftrightarrow Convention:

$$\text{tr}(T^a T^b) = \frac{1}{2} \delta_{ab}. \quad (7.11)$$

$\Rightarrow f^{abc}$ are totally antisymmetric, since

$$f^{abc} = -2i \text{tr}((T^b T^c - T^c T^b) T^a) = -2i \text{tr}(T^b T^c T^a - T^c T^b T^a) = \text{cyclic in } abc.$$

- Finite-dimensional representations are unitary:

$$D(\omega)^\dagger = \exp(i\omega^a T^{a\dagger}) \stackrel{!}{=} D(\omega)^{-1} = \exp(i\omega^a T^a), \quad \text{i.e. } T^a = T^{a\dagger}. \quad (7.12)$$

\Rightarrow Generators T^a are hermitian.

Comment:

Defining representations of matrix Lie algebras:

$T^a = N \times N$ matrices with special properties:

- $GL(N, \mathbb{C})$: complex, no restriction
- $SL(N, \mathbb{C})$: complex, traceless
- $SO(N)$: imaginary, antisymmetric
- $SU(N)$: hermitian, traceless

Example: $SU(2)$ = relevant group for angular momentum in QM

- Group elements in the defining (*fundamental*, i.e. lowest-dimensional) representation:
 U = unitary 2×2 matrix with $\det U = 1$.

- Generators in the *fundamental representation*:

T^a = traceless, hermitian 2×2 matrices ($a = 1, 2, 3$).

Usual convention: $T^a = \frac{\sigma^a}{2}$, σ^a = Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (7.13)$$

Lie algebra:

$$[\sigma^a, \sigma^b] = 2i\epsilon^{abc}\sigma^c, \quad \text{structure constants} = \epsilon^{abc} = \text{totally antisym. } \epsilon\text{-tensor} \quad (7.14)$$

- Finite group elements in fundamental representation:

$$U(\vec{\omega}) = \exp\left(-\frac{i}{2}\vec{\omega} \cdot \vec{\sigma}\right) = \cos\left(\frac{\omega}{2}\right) - i\vec{e} \cdot \vec{\sigma} \sin\left(\frac{\omega}{2}\right), \quad (7.15)$$

where $\vec{\omega} = \omega\vec{e}$ ($\vec{e}^2 = 1$).

- Adjoint representation:

$$T_{\text{adj}}^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad T_{\text{adj}}^2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \quad T_{\text{adj}}^3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (7.16)$$

Finite group elements:

$R(\vec{\omega}) = \exp(-iT_{\text{adj}}^a \omega^a)$ = 3dim. rotation matrices with angle ω around axis \vec{e} .

- Any representation: $T^a = J^a$ = components of angular momentum.

7.1.3 Irreducible representations

Definitions:

- A representation is called *reducible* if there is a subspace H of V that is invariant under all matrices $D(g)$, i.e. if all $D(g)$ can be written in the block form

$$D(g) = \begin{pmatrix} D_1(g) & E(g) \\ 0 & D_2(g) \end{pmatrix}. \quad (7.17)$$

- If there is no invariant subspace, the representation is called *irreducible*.

- A representation is called *fully reducible* if all $D(g)$ can be written in block-diagonal form,

$$D(g) = \begin{pmatrix} D_1(g) & 0 & 0 & \dots \\ 0 & D_2(g) & 0 & \dots \\ & & \ddots & \\ 0 & \dots & & D_n(g) \end{pmatrix}, \quad (7.18)$$

where the D_n are irreducible, i.e. a fully reducible representation is the direct sum of irreducible representations:

$$D = D_1 \oplus D_2 \oplus \dots \oplus D_n. \quad (7.19)$$

- Definitions of (ir)reducibility for Lie algebras analogously.
- An operator C commuting with all elements of the Lie algebra is called *Casimir operator*:

$$[C, T^a] = 0. \quad (7.20)$$

Some facts about (ir)reducibility:

- Irreducible representations of abelian groups are one-dimensional.
- All unitary reducible group representations are fully reducible.
- *Schur's Lemma*:

If a linear mapping A on a vector space V commutes with all matrices $D(g)$ of an irreducible representation of the group G on V , i.e.

$$AD(g) = D(g)A \quad (7.21)$$

for all $g \in G$, then A is a multiple of the identity:

$$A = \lambda_D \mathbf{1}, \quad (7.22)$$

where λ_D depends on the representation.

- Schur's Lemma applied to Lie algebras:
Casimir operators $C \propto \mathbf{1}$ in an irreducible representation.
- In Lie algebras with $f^{abc} =$ totally antisymmetric (e.g. for compact Lie groups) there is always the quadratic Casimir operator,

$$C_2 = T^a T^a. \quad (7.23)$$

||| Comment:

||| Proof that C_2 satisfies (7.20):

$$[C_2, T^b] = T^a [T^a, T^b] + [T^a, T^b] T^a = i f^{abc} (T^a T^c + T^c T^a) = 0. \quad (7.24)$$

Example: irreducible representations of $SU(2)$

↔ known from the angular momentum in QM

- Quadratic Casimir operator:

$$\vec{J}^2 = \sum_a J^a J^a = \text{total angular momentum operator}, \quad [\vec{J}^2, J^a] = 0. \quad (7.25)$$

⇒ Diagonalization of \vec{J}^2 and one component J^a possible, usual choice J^3 .

- Irreducible representation $D^{(j)}$ for each fixed value of $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots$:

$$\begin{aligned} \vec{J}^2 |j, m\rangle &= j(j+1) |j, m\rangle, \\ J^3 |j, m\rangle &= m |j, m\rangle, \quad m = -j, -j+1, \dots, j. \end{aligned} \quad (7.26)$$

$\{|j, m\rangle\} = (2j+1)$ -dimensional multiplet for each fixed j .

7.1.4 Constructing representations

New group representations from two representations D_i ($i = 1, 2$) on vector spaces V_i ($\dim V_i = n_i$):

- $D_1 \oplus D_2$ on the *direct sum* $V_1 \oplus V_2$ of vector spaces ($\dim = n_1 + n_2$):

$$(D_1 \oplus D_2)(g) = \begin{pmatrix} D_1(g) & 0 \\ 0 & D_2(g) \end{pmatrix}, \quad v_1 \oplus v_2 = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \quad v_i \in V_i,$$

$$\text{i.e. } (D_1 \oplus D_2)(g)(v_1 \oplus v_2) = (D_1(g)v_1) \oplus (D_2(g)v_2). \quad (7.27)$$

Representation is reducible by construction.

- $D_1 \otimes D_2$ on the *direct product* $V_1 \otimes V_2$ of vector spaces ($\dim = n_1 n_2$):

$$(D_1 \otimes D_2)(g)(v_1 \otimes v_2) = (D_1(g)v_1) \otimes (D_2(g)v_2). \quad (7.28)$$

Representation is in general reducible, but decomposable into irreducible blocks $D^{(i)}$:

$$D_1 \otimes D_2 = D^{(i_1)} \oplus \dots \oplus D^{(i_n)}. \quad (7.29)$$

Definitions carry over to Lie algebras: $D(g) = \mathbf{1} - i\omega_a T_D^a + \dots$

- Direct sum representation on $V_1 \oplus V_2$:

$$T_{D_1 \oplus D_2}^a = \begin{pmatrix} T_{D_1}^a & 0 \\ 0 & T_{D_2}^a \end{pmatrix}, \quad T_{D_1 \oplus D_2}^a(v_1 \oplus v_2) = (T_{D_1}^a v_1) \oplus (T_{D_2}^a v_2). \quad (7.30)$$

- Direct product representation on $V_1 \otimes V_2$:

$$T_{D_1 \otimes D_2}^a(v_1 \otimes v_2) = (T_{D_1}^a v_1) \otimes v_2 + v_1 \otimes (T_{D_2}^a v_2). \quad (7.31)$$

Example: product representations of $SU(2)$

\hookrightarrow addition of two angular momenta \vec{J}_i ($i = 1, 2$) with respective multiplets $|j_i, m_i\rangle$:

$$\vec{J}|j_1, m_1\rangle \otimes |j_2, m_2\rangle = (\vec{J}_1|j_1, m_1\rangle) \otimes |j_2, m_2\rangle + |j_1, m_1\rangle \otimes (\vec{J}_2|j_2, m_2\rangle). \quad (7.32)$$

Decomposition into irreducible blocks: (Clebsch–Gordan series)

$$|j_1, m_1\rangle \otimes |j_2, m_2\rangle = \sum_j c_{j,m} |j, m = m_1 + m_2\rangle \quad \text{with} \quad |j_1 - j_2| \leq j \leq j_1 + j_2, \quad (7.33)$$

in terms of representation spaces:

$$D^{(j_1)} \otimes D^{(j_2)} = D^{(|j_1 - j_2|)} \oplus D^{(|j_1 - j_2| + 1)} \oplus \dots \oplus D^{(j_1 + j_2)}. \quad (7.34)$$

Specifically:

$$\begin{aligned} D^{(\frac{1}{2})} \otimes D^{(\frac{1}{2})} &= D^{(0)} \oplus D^{(1)}, \\ D^{(1)} \otimes D^{(1)} &= D^{(0)} \oplus D^{(1)} \oplus D^{(2)}, \quad \text{etc.} \end{aligned} \quad (7.35)$$

7.2 Irreducible representations of the Lorentz group

Recall: Lorentz transformations and generators (see Chap. 2)

$$\Lambda = \exp \left\{ -i \left(\underbrace{\nu^k K^k}_{\text{boost}} + \underbrace{\varphi^k J^k}_{\text{rotation}} \right) \right\}. \quad (7.36)$$

Lie algebra of generators:

$$[J^i, J^j] = i\epsilon^{ijk} J^k, \quad (7.37)$$

$$[J^i, K^j] = i\epsilon^{ijk} K^k, \quad (7.38)$$

$$[K^i, K^j] = -i\epsilon^{ijk} J^k. \quad (7.39)$$

Simplification by change of basis:

$$T_{1,2}^k = \frac{1}{2}(J^k \mp iK^k). \quad \Rightarrow \quad [T_a^i, T_b^j] = i\epsilon^{ijk} T_a^k \delta_{ab}. \quad (7.40)$$

\Rightarrow Lie algebras of L_+^\uparrow and $SU(2) \otimes SU(2)$ closely related (complex versions are identical).

Construction of irreducible representations of L_+^\uparrow : (analogy to $SU(2)$ case)

Two commuting generators: T_1^3, T_2^3 ; two Casimir operators: \vec{T}_1^2, \vec{T}_2^2 .

\hookrightarrow Multiplets $|j_1, m_1; j_2, m_2\rangle \equiv |j_1, m_1\rangle_1 \otimes |j_2, m_2\rangle_2$ span $(2j_1 + 1)(2j_2 + 1)$ -dimensional irreducible representation $D^{(j_1, j_2)}$ for fixed j_1, j_2 :

$$\begin{aligned} T_a^3 |j_1, m_1; j_2, m_2\rangle &= m_a |j_1, m_1; j_2, m_2\rangle, \quad m_a = -j_a, -j_a + 1, \dots, j_a, \\ \vec{T}_a^2 |j_1, m_1; j_2, m_2\rangle &= j_a(j_a + 1) |j_1, m_1; j_2, m_2\rangle, \quad j_a = 0, \frac{1}{2}, 1, \dots \end{aligned} \quad (7.41)$$

Lorentz transformations in $D^{(j_1, j_2)}$:

$$\Lambda^{(j_1, j_2)} = \exp\left(-i(\vec{\varphi} + i\vec{\nu})\vec{T}_1^{(j_1)}\right) \exp\left(-i(\vec{\varphi} - i\vec{\nu})\vec{T}_2^{(j_2)}\right), \quad [T_1^k, T_2^l] = 0. \quad (7.42)$$

Comments:

- Hermitian $SU(2)$ generators $T_a^{(j_a)}$ constructed as in non-relativistic QM.
- Angular momentum $\vec{J} = \vec{T}_1 + \vec{T}_2$.
 $\hookrightarrow D^{(j_1, j_2)}$ contains angular momenta $j = |j_1 - j_2|, |j_1 - j_2| + 1, \dots, j_1 + j_2$.
- $\Lambda^{(j_1, j_2)}$ is unitary only for pure rotations ($\vec{\nu} = 0$).

Parity transformation:

Behaviour of generators: $P: \vec{J} \rightarrow \vec{J}$ (pseudo-vector),
 $P: \vec{K} \rightarrow -\vec{K}$ (vector).

$\Rightarrow P$ interchanges the two $SU(2)$ factors:

$$\vec{T}_1 \xleftrightarrow{P} \vec{T}_2, \quad \text{i.e. } P: D^{(j_1, j_2)} \rightarrow D^{(j_2, j_1)}. \quad (7.43)$$

$\Rightarrow P$ -invariant representations: $D^{(j, j)}$ and $D^{(j_1, j_2)} \oplus D^{(j_2, j_1)}$ for $j_1 \neq j_2$.

7.3 Fundamental spinor representations

- $D^{(\frac{1}{2}, 0)}$: *Right-chiral fundamental representation*

$$\text{Generators: } T_1^{(\frac{1}{2}), i} = \frac{\sigma^i}{2}, \quad T_2^{0, i} = 0. \quad (7.44)$$

$$\text{Transformations: } \Lambda^{(\frac{1}{2}, 0)} = \exp\left(-\frac{i}{2}(\vec{\varphi} + i\vec{\nu})\vec{\sigma}\right) \equiv \Lambda_R. \quad (7.45)$$

$$\text{Multiplets: } \chi = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}, \quad \text{right-chiral Weyl spinors.} \quad (7.46)$$

- $D^{(0, \frac{1}{2})}$: *Left-chiral fundamental representation*

$$\text{Generators: } T_1^{0, i} = 0, \quad T_2^{(\frac{1}{2}), i} = \frac{\sigma^i}{2}. \quad (7.47)$$

$$\text{Transformations: } \Lambda^{(0, \frac{1}{2})} = \exp\left(-\frac{i}{2}(\vec{\varphi} - i\vec{\nu})\vec{\sigma}\right) \equiv \Lambda_L. \quad (7.48)$$

$$\text{Multiplets: } \bar{\chi} = \begin{pmatrix} \bar{\chi}_1 \\ \bar{\chi}_2 \end{pmatrix}, \quad \text{left-chiral Weyl spinors.} \quad (7.49)$$

Properties of $\Lambda_{R,L}$:

- $\Lambda_{R,L}$ = complex 2×2 matrices with $\det \Lambda_{R,L} = 1$, i.e. $\Lambda_{R,L} \in SL(2, \mathbb{C})$.
- Useful identities:

$$\Lambda_R^\dagger = \Lambda_L^{-1}, \quad \Lambda_L^\dagger = \Lambda_R^{-1}. \quad (7.50)$$

- Relation by complex conjugation:

$$\begin{aligned} \epsilon^{-1} \sigma^i \epsilon &= -\sigma^{i*}, & \epsilon &= i\sigma^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \\ \Rightarrow \Lambda_R^* &= \exp\left(\frac{i}{2}(\vec{\varphi} - i\vec{\nu})\vec{\sigma}^*\right) = \exp\left(-\frac{i}{2}\epsilon^{-1}(\vec{\varphi} - i\vec{\nu})\vec{\sigma}\epsilon\right) = \epsilon^{-1}\Lambda_L\epsilon. \end{aligned} \quad (7.51)$$

$$\Rightarrow \text{Equivalence:} \quad \left(D^{(\frac{1}{2},0)}\right)^* \sim D^{(0,\frac{1}{2})}, \quad \text{i.e.} \quad D^{(\frac{1}{2},0)} \xleftrightarrow{\text{complex conjugation}} D^{(0,\frac{1}{2})}.$$

Comment:

Construction of left (right) spinors from a right (left) spinors:

$$\begin{aligned} \chi \in D^{(\frac{1}{2},0)} : & \quad (\epsilon\chi^*) \rightarrow \epsilon\Lambda_R^*\chi^* = \Lambda_L(\epsilon\chi^*), \quad \text{i.e.} \quad \epsilon\chi^* \in D^{(0,\frac{1}{2})}, \\ \bar{\chi} \in D^{(0,\frac{1}{2})} : & \quad (\epsilon^{-1}\bar{\chi}^*) \rightarrow \epsilon^{-1}\Lambda_L^*\bar{\chi}^* = \Lambda_R(\epsilon^{-1}\bar{\chi}^*), \quad \text{i.e.} \quad (\epsilon^{-1}\bar{\chi}^*) \in D^{(\frac{1}{2},0)}. \end{aligned} \quad (7.52)$$

Dirac spinors

Parity-invariant representation for spin- $\frac{1}{2}$ fermions: (e.g. needed for electromagnetism)

- *Dirac representation:*

$$D^{(\frac{1}{2},0)} \oplus D^{(0,\frac{1}{2})} \quad (\text{irreducible under } L_+^\dagger \otimes P) \quad (7.53)$$

- *Dirac spinor* ψ :

$$\psi = \begin{pmatrix} \chi \\ \bar{\xi} \end{pmatrix}. \quad (7.54)$$

Lorentz transformation of ψ :

$$\psi \xrightarrow{\Lambda} \psi' = \begin{pmatrix} \Lambda_R \chi \\ \Lambda_L \bar{\xi} \end{pmatrix} = \underbrace{\begin{pmatrix} \Lambda_R & 0 \\ 0 & \Lambda_L \end{pmatrix}}_{\equiv S(\Lambda)} \psi = S(\Lambda)\psi. \quad (7.55)$$

7.4 Product representations

Decomposition of products $D^{(j_1, j_2)} \otimes D^{(j'_1, j'_2)}$ into irreducible blocks:

\hookrightarrow Possible upon using relations from $SU(2)$:

- From Clebsch–Gordan series of $SU(2)$:

$$\begin{aligned} D^{(j,0)} \otimes D^{(j',0)} &= D^{(j+j',0)} \oplus D^{(j+j'-1,0)} \oplus \dots \oplus D^{(|j-j'|,0)}, \\ D^{(0,j)} \otimes D^{(0,j')} &\text{ analogously.} \end{aligned} \quad (7.56)$$

- Independence of $SU(2)$ factors, i.e. $[T_1^k, T_2^l] = 0$:

$$D^{(j_1, j_2)} = \underbrace{D^{(j_1,0)} \otimes D^{(0, j_2)}}_{\text{independent factors}}. \quad (7.57)$$

$$\begin{aligned} \Rightarrow D^{(j_1, j_2)} \otimes D^{(j'_1, j'_2)} &= D^{(j_1,0)} \otimes D^{(0, j_2)} \otimes D^{(j'_1,0)} \otimes D^{(0, j'_2)} \\ &= \underbrace{D^{(j_1,0)} \otimes D^{(j'_1,0)}}_{\text{use } SU(2) \text{ relation (7.56)}} \otimes \underbrace{D^{(0, j_2)} \otimes D^{(0, j'_2)}}_{\text{use } SU(2) \text{ relation (7.56)}} \\ &= \dots = D^{(j_1+j'_1, j_2+j'_2)} \oplus D^{(j_1+j'_1-1, j_2+j'_2)} \oplus D^{(j_1+j'_1, j_2+j'_2-1)} \dots \oplus D^{(|j_1-j'_1|, |j_2-j'_2|)}. \end{aligned} \quad (7.58)$$

Note: Reduction important in construction of covariant quantities from products of multiplets (e.g. invariants for Lagrangians).

Examples:

- *Invariant spinor product:*

$$D^{(\frac{1}{2},0)} \otimes D^{(\frac{1}{2},0)} = D^{(1,0)} \oplus \underbrace{D^{(0,0)}}_{\substack{\text{trivial representation} \\ \text{(objects Lorentz invariant)}}} \quad (7.59)$$

$\Rightarrow \exists 2 \times 2$ matrix $A = (a_{ij})$ so that

$$a_{ij} \chi'_i \xi'_j = a_{ij} (\Lambda_R)_{ik} (\Lambda_R)_{jl} \chi_k \xi_l \stackrel{!}{=} a_{ij} \chi_i \xi_j = \text{invariant}, \quad \text{i.e. } \Lambda_R^T A \Lambda_R = A \quad \forall \Lambda_R \in L. \quad (7.60)$$

Solution: $A = \epsilon =$ totally antisymmetric tensor.

\Rightarrow Lorentz-invariant product of Weyl spinors: (left-handed case analogously)

$$\langle \chi \xi \rangle \equiv \epsilon_{ij} \chi_i \xi_j, \quad \langle \bar{\chi} \bar{\xi} \rangle \equiv \epsilon_{ij} \bar{\chi}_i \bar{\xi}_j. \quad (7.61)$$

- *4-vector representation:*

Required: real, P -invariant representation that contains spin value $j = 1$ (vector!).

Simplest candidate:

$$D^{(\frac{1}{2}, \frac{1}{2})} = D^{(\frac{1}{2}, 0)} \otimes D^{(0, \frac{1}{2})} \sim D^{(\frac{1}{2}, 0)} \otimes \left(D^{(\frac{1}{2}, 0)} \right)^* \quad (7.62)$$

To show: $\exists 2 \times 2$ matrices C^μ so that

$$\chi_i^\dagger C_{ij}^\mu \xi_j = 4\text{-vector}, \quad \text{i.e.} \quad \Lambda_R^\dagger C^\mu \Lambda_R = \Lambda^\mu{}_\nu C^\nu. \quad (7.63)$$

Solution: (see Exercise 8.3)

$$C^\mu = \sigma^\mu = (\mathbf{1}, \sigma^1, \sigma^2, \sigma^3). \quad (7.64)$$

Analogously:

$$\Lambda_L^\dagger \bar{\sigma}^\mu \Lambda_L = \Lambda^\mu{}_\nu \bar{\sigma}^\nu, \quad \bar{\sigma}^\mu = (\mathbf{1}, -\sigma^1, -\sigma^2, -\sigma^3). \quad (7.65)$$

\Rightarrow 4-vectors from products of Weyl spinors:

$$\chi^\dagger \sigma^\mu \xi, \quad \bar{\chi}^\dagger \bar{\sigma}^\mu \bar{\xi}. \quad (7.66)$$

- *Dirac representation:*

Covariants from products of two Dirac spinors ψ_1, ψ_2 ?

$$\begin{aligned} & \left[D^{(\frac{1}{2}, 0)} \oplus D^{(0, \frac{1}{2})} \right] \otimes \left[D^{(\frac{1}{2}, 0)} \oplus D^{(0, \frac{1}{2})} \right] \\ &= \underbrace{D^{(0, 0)}}_{\text{scalar}} \oplus \underbrace{D^{(0, 0)}}_{\text{pseudo-scalar}} \oplus \underbrace{D^{(\frac{1}{2}, \frac{1}{2})}}_{\text{vector}} \oplus \underbrace{D^{(\frac{1}{2}, \frac{1}{2})}}_{\text{pseudo-vector}} \oplus \underbrace{D^{(1, 0)} \oplus D^{(0, 1)}}_{\text{anti-sym. rank-2 tensors}}. \end{aligned} \quad (7.67)$$

Auxiliary quantities for explicit construction:

$$\gamma^\mu \equiv \begin{pmatrix} 0 & \bar{\sigma}^\mu \\ \sigma^\mu & 0 \end{pmatrix} \quad \text{Dirac matrices in chiral representation}, \quad (7.68)$$

$$\bar{\psi} \equiv \psi^\dagger \gamma_0 = (\bar{\phi}^\dagger, \chi^\dagger) \quad \text{adjoint Dirac spinor to } \psi = \begin{pmatrix} \chi \\ \bar{\phi} \end{pmatrix}, \quad (7.69)$$

$$\gamma_5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = -\frac{i}{4!}\epsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}. \quad (7.70)$$

\Rightarrow Construction of covariants: (see Exercise 9.3)

$$\bar{\psi}_1 \psi_2 = \text{scalar}, \quad (7.71)$$

$$\bar{\psi}_1 \gamma_5 \psi_2 = \text{pseudo-scalar}, \quad (7.72)$$

$$\bar{\psi}_1 \gamma^\mu \psi_2 = \text{vector}, \quad (7.73)$$

$$\bar{\psi}_1 \gamma^\mu \gamma_5 \psi_2 = \text{pseudo-vector}, \quad (7.74)$$

$$\bar{\psi}_1 \gamma^\mu \gamma^\nu \psi_2 = \text{rank-2 tensor}, \quad (7.75)$$

$$\bar{\psi}_1 \gamma^\mu \gamma^\nu \gamma_5 \psi_2 = \text{rank-2 pseudo-tensor}. \quad (7.76)$$

Some properties of the Dirac matrices: (see Exercise 9.2)

$$\bullet \quad \{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}, \quad \gamma_0 \gamma^\mu \gamma_0 = (\gamma^\mu)^\dagger, \quad \{\gamma^\mu, \gamma^5\} = 0, \quad (7.77)$$

$$\bullet \quad \text{Tr}[\gamma^\mu \gamma^\nu] = 4g^{\mu\nu}, \quad \text{Tr}[\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma] = 4[g^{\mu\nu} g^{\rho\sigma} - g^{\mu\rho} g^{\nu\sigma} + g^{\mu\sigma} g^{\nu\rho}], \quad (7.78)$$

$$\text{Tr}[\gamma^{\mu_1} \dots \gamma^{\mu_{2n+1}}] = 0, \quad n = 0, 1, \dots, \quad (7.79)$$

$$\bullet \quad \text{Tr}[\gamma_5] = \text{Tr}[\gamma^\mu \gamma^\nu \gamma_5] = 0, \quad \text{Tr}[\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma_5] = -4i\epsilon^{\mu\nu\rho\sigma}, \quad (7.80)$$

$$\bullet \quad \gamma^\alpha \gamma_\alpha = 4, \quad \gamma^\alpha \gamma^\mu \gamma_\alpha = -2\gamma^\mu. \quad (7.81)$$

7.5 Relativistic wave equations

7.5.1 Relativistic fields

Requirement by relativistic covariance:

Fields $\Phi_k(x)$ describing a specific particle transform in an *irreducible* representation of L_+^\uparrow or $L_+^\uparrow \times P$ (reducible case: different irreducible blocks \Rightarrow different particles).

Lorentz transformation: (Φ = classical field, no operator)

$$\Phi'_k(x') = \underbrace{S_{kl}(\Lambda)}_{\substack{\text{transformation matrix in irreducible} \\ \text{representation of } L_+^\uparrow \text{ or } L_+^\uparrow \times P}} \Phi_l(x), \quad x' = \Lambda x. \quad (7.82)$$

$$\Phi'(x) = \underbrace{S(\Lambda)}_{\substack{\text{transformation of} \\ \text{inner degrees of} \\ \text{freedom} \rightarrow \text{spin}}} \underbrace{\Phi(\Lambda^{-1}x)}_{\substack{\text{transformation of} \\ \text{space-time argument} \\ \Leftrightarrow \text{orbital angular momentum}}} \quad (7.83)$$

Transformations in exponential form:

$$S(\Lambda) = \exp\left\{-\frac{i}{2}\omega_{\alpha\beta}M^{\alpha\beta}\right\} = \text{finite-dim. representation}, \quad (7.84)$$

$$\Phi(\Lambda^{-1}x) = \underbrace{\exp\left\{-\frac{i}{2}\omega_{\alpha\beta}L^{\alpha\beta}\right\}}_{\substack{\text{differential operator} \\ L^{\alpha\beta} = x^\alpha \hat{p}^\beta - x^\beta \hat{p}^\alpha \\ = \text{generalized orbital angular mom.}}} \Phi(x). \quad (7.85)$$

$$\Rightarrow \Phi'(x) = \exp\left\{-\frac{i}{2}\omega_{\alpha\beta} \underbrace{(M^{\alpha\beta} + L^{\alpha\beta})}_{\rightarrow \text{total angular momentum}}\right\} \Phi(x). \quad (7.86)$$

7.5.2 Relativistic wave equations for free particles

Basic requirements:

- Qm. superposition principle \rightarrow *linearity* of differential equation

General ansatz:

$$\underbrace{\Pi_{kl}(m, i\partial^\mu)}_{\substack{N \times N\text{-matrix-valued} \\ \text{differential operator}}} \Phi_l(x) = 0, \quad (7.87)$$

where m = particle mass, N determined by particle spin.

Order of differential eq. ≤ 2 (otherwise strange behaviour of solutions).

- *Covariance*: (7.87) has to imply

$$0 \stackrel{!}{=} \Pi(m, i\Lambda\partial) \Phi'(\Lambda x) \quad \text{with} \quad \Phi'(\Lambda x) = S(\Lambda) \Phi(x). \quad (7.88)$$

$$\Rightarrow \Pi(m, i\Lambda\partial) \stackrel{!}{=} \underbrace{S'(\Lambda)}_{\text{could also be in another representation}} \Pi(m, i\partial) S(\Lambda^{-1}). \quad (7.89)$$

- *Mass-shell condition*:

In momentum space only field modes with $p^2 = m^2$ should contribute to solution:

$$\begin{aligned} \Phi(x) &= \int \underbrace{d\tilde{p}}_{\substack{d^4p \\ (2\pi)^4}} e^{-ixp} \tilde{\Phi}(\vec{p}). \\ &= \frac{d^4p}{(2\pi)^4} (2\pi)\delta(p^2 - m^2)\theta(p_0) \end{aligned} \quad (7.90)$$

- *Spin projection*:

Operator Π should project onto genuine spin- j part of representation $D^{(j_1, j_2)}$, where $|j_1 - j_2| \leq j \leq j_1 + j_2$. If Π does not perform this projection, additional constraints are needed to achieve it.

Examples:

- Klein–Gordon equation: $j = 0$.

$$(\square + m^2)\phi(x) = 0, \quad \Pi(m, i\partial) = \partial^\mu \partial_\mu + m^2, \quad S(\Lambda) = 1, \quad (7.91)$$

i.e. requirements trivially fulfilled.

- Maxwell equation for elmg. 4-vector potential A^μ in Lorenz gauge:

$$\square A^\mu = 0, \quad \Pi(i\partial) = \partial^\mu \partial_\mu, \quad S(\Lambda) = \Lambda. \quad (7.92)$$

Lorenz condition $\partial^\mu A_\mu = 0$ eliminates spin-0 part in $D^{(\frac{1}{2}, \frac{1}{2})}$ representation, only spin-1 part remains.

|| Comment:
 || Photons do not really carry spin 1, but helicity $h = \pm 1$ (=spin projected to direction
 || of flight), see below.

7.5.3 The Dirac equation

Wave equations for spin- $\frac{1}{2}$ fields:

Minimalistic attempt:

Wave equations for $\chi(x) \in D^{(\frac{1}{2},0)}$ and $\bar{\phi}(x) \in D^{(0,\frac{1}{2})}$ (smallest representations with $j = \frac{1}{2}$):

- Non-trivial transformation property of $\chi, \bar{\phi}$ should result from wave equation (otherwise additional constraints needed).
 $\hookrightarrow \Pi(m, i\partial)$ should mix field components (i.e. KG operator not acceptable).
- Relevant covariant objects for wave equations:

$$\chi, \bar{\sigma}^\mu \partial_\mu \bar{\phi} \in D^{(\frac{1}{2},0)}, \quad \bar{\phi}, \sigma^\mu \partial_\mu \chi \in D^{(0,\frac{1}{2})}. \quad (7.93)$$

Proof:

$$\chi(x) \rightarrow \chi'(x') = \Lambda_R \chi(x) \quad (7.94)$$

$$\begin{aligned} \sigma^\mu \partial_\mu \chi(x) &\rightarrow \sigma^\mu \partial'_\mu \chi'(x') = \sigma^\mu \Lambda_\mu{}^\nu \partial_\nu \Lambda_R \chi(x) \\ &= \Lambda_L \underbrace{(\Lambda_L^{-1} \sigma^\mu \Lambda_R)}_{\Lambda^\dagger \sigma^\mu \Lambda_R} \Lambda_\mu{}^\nu \partial_\nu \chi(x) = \Lambda_L \sigma^\mu \partial_\mu \chi(x). \\ &= \Lambda_R^\dagger \sigma^\mu \Lambda_R = \Lambda^\mu{}_\rho \sigma^\rho \end{aligned} \quad (7.95)$$

$$\bar{\sigma}^\mu \partial_\mu \bar{\phi} \text{ analogously.} \quad \text{q.e.d.}$$

- Consequence: Only possibility for separate wave equations for $\chi, \bar{\phi}$:

$$\sigma^\mu \partial_\mu \chi = 0, \quad \bar{\sigma}^\mu \partial_\mu \bar{\phi} = 0. \quad \text{Weyl equations} \quad (7.96)$$

Note: $\sigma^\mu \partial_\mu \chi = c\chi$, etc., not compatible with relativistic covariance !

- Solution of Weyl equations:

Fourier ansatz: $\chi(x) = e^{\pm ikx} n_R$ with $n_R = \begin{pmatrix} n_{R,1} \\ n_{R,2} \end{pmatrix}$.

$$\begin{aligned} \hookrightarrow 0 &\stackrel{!}{=} \underbrace{\begin{pmatrix} k^0 - k^3 & -k^1 + ik^2 \\ -k^1 - ik^2 & k^0 + k^3 \end{pmatrix}}_{\det(\dots) = (k^0)^2 - (k^1)^2 - (k^2)^2 - (k^3)^2 = k^2} \begin{pmatrix} n_{R,1} \\ n_{R,2} \end{pmatrix}. \end{aligned} \quad (7.97)$$

\Rightarrow Non-trivial solutions only for $k^2 = 0$, i.e. Weyl fermions are massless !

Explicitly:

$$k^\mu = k^0(1, \vec{e}), \quad \vec{e} = \begin{pmatrix} \cos \varphi \sin \theta \\ \sin \varphi \sin \theta \\ \cos \theta \end{pmatrix} \Rightarrow n_R = \begin{pmatrix} e^{-i\varphi} \cos \frac{\theta}{2} \\ \sin \frac{\theta}{2} \end{pmatrix}. \quad (7.98)$$

Analogously:

$$\bar{\phi}(x) = e^{\pm ikx} n_L, \quad k^2 = 0, \quad n_L = \begin{pmatrix} \sin \frac{\theta}{2} \\ -e^{i\varphi} \cos \frac{\theta}{2} \end{pmatrix}. \quad (7.99)$$

The Dirac equation:

Combine Weyl spinors $\chi, \bar{\phi}$ to Dirac spinor $\psi = \begin{pmatrix} \chi \\ \bar{\phi} \end{pmatrix}$.

\Leftrightarrow Two covariant 1st-order equations possible:

$$i\sigma^\mu \partial_\mu \chi = c_1 \bar{\phi}, \quad i\bar{\sigma}^\mu \partial_\mu \bar{\phi} = c_2 \chi. \quad (7.100)$$

Note: By appropriate rescaling, equality $c_1 = c_2 = m$ can be achieved.
(Identification of m as mass later.)

Matrix form:

$$i \underbrace{\begin{pmatrix} 0 & \bar{\sigma}^\mu \\ \sigma^\mu & 0 \end{pmatrix}}_{= \gamma^\mu, \text{ Dirac matrices}} \partial_\mu \begin{pmatrix} \chi \\ \bar{\phi} \end{pmatrix} - m \begin{pmatrix} \chi \\ \bar{\phi} \end{pmatrix} = 0. \quad (7.101)$$

$$\Rightarrow (i\gamma^\mu \partial_\mu - m) \psi = 0. \quad \text{Dirac Equation} \quad (7.102)$$

Notation: $\not{a} \equiv \gamma^\mu a_\mu = \gamma_\mu a^\mu$ (*Feynman dagger*) \Rightarrow Dirac eq.: $(i\not{\partial} - m) \psi = 0$.

Covariance: (see Exercise 9.3)

$$\psi(x) \rightarrow \psi'(x') = S(\Lambda) \psi(x), \quad S(\Lambda) = \begin{pmatrix} \Lambda_R & 0 \\ 0 & \Lambda_L \end{pmatrix}, \quad (7.103)$$

$$\not{a} \rightarrow \not{a}' = \gamma_\mu \Lambda^\mu{}_\nu a^\nu = S(\Lambda) \not{a} S(\Lambda)^{-1}. \quad (7.104)$$

$$\begin{aligned} \Rightarrow (i\not{\partial} - m) \psi(x) = 0 &\rightarrow (i\not{\partial}' - m) \psi'(x') \\ &= S(\Lambda) (i\not{\partial} - m) S(\Lambda)^{-1} S(\Lambda) \psi(x) \\ &= S(\Lambda) (i\not{\partial} - m) \psi(x) = 0. \end{aligned} \quad (7.105)$$

Chapter 8

Free Dirac fermions

8.1 Solutions of the classical Dirac equation

Dirac eq.: $(i\cancel{\partial} - m)\psi = 0$.

Note: Each component of ψ obeys the KG equation:

$$\begin{aligned} 0 &= (-i\cancel{\partial} - m)(i\cancel{\partial} - m)\psi = (\underbrace{\cancel{\partial}^2}_{\square} + m^2)\psi = (\square + m^2)\psi. \\ &= \gamma^\mu \gamma^\nu \partial_\mu \partial_\nu = \frac{1}{2}\{\gamma^\mu, \gamma^\nu\} \partial_\mu \partial_\nu = g^{\mu\nu} \partial_\mu \partial_\nu = \partial^2 \end{aligned} \quad (8.1)$$

Fourier ansatz:

$$\psi(x) = e^{-ikx} u(k), \quad u(k) = \text{constant 4-component Dirac spinor.} \quad (8.2)$$

- Eq. (8.1) implies $k^2 = m^2$. \rightarrow Set $k_0 = \sqrt{\vec{k}^2 + m^2}$.
- Ansatz leads to Dirac eq. in momentum space:

$$(\cancel{k} - m)u(k) = 0. \quad (8.3)$$

\hookrightarrow 4-dim. system of linear equations.

Solution of Eq. (8.3) in two steps:

1. Solve equation first in rest frame of k^μ , i.e. for $k_r^\mu = (m, \vec{0})$:

$$k^\mu = (k_0, \vec{k}) = \Lambda^\mu{}_\nu k_r^\nu, \quad \vec{k} = |\vec{k}| \vec{e}, \quad \vec{e} = \begin{pmatrix} \cos \varphi \sin \theta \\ \sin \varphi \sin \theta \\ \cos \theta \end{pmatrix}. \quad (8.4)$$

Using the chiral representation for γ^μ yields

$$0 = (\cancel{k}_r - m)u(k_r) = m(\gamma_0 - \mathbf{1})u(k_r) = m \begin{pmatrix} -\mathbf{1} & +\mathbf{1} \\ +\mathbf{1} & -\mathbf{1} \end{pmatrix} u(k_r). \quad (8.5)$$

\hookrightarrow Two independent solutions of the block form $u(k_r) = \begin{pmatrix} n \\ n \end{pmatrix}$, e.g. $n = n_{R,L}$.

2. Boost k_r^μ into original system:

$$u(k) = S(\Lambda) u(k_r) = \begin{pmatrix} \Lambda_R & 0 \\ 0 & \Lambda_L \end{pmatrix} u(k_r), \quad (8.6)$$

$$\text{with } \Lambda_R = \exp\left\{\frac{\nu}{2}\vec{e}\vec{\sigma}\right\}, \quad \Lambda_L = \exp\left\{-\frac{\nu}{2}\vec{e}\vec{\sigma}\right\}, \quad \nu = \frac{1}{2} \ln\left(\frac{k_0 + |\vec{k}|}{k_0 - |\vec{k}|}\right) = \text{rapidity.}$$

Simple form of $\Lambda_{R,L}$ after diagonalizing $\vec{e}\vec{\sigma} = U\sigma^3U^\dagger$:

$$\Lambda_R = U \begin{pmatrix} e^{\nu/2} & 0 \\ 0 & e^{-\nu/2} \end{pmatrix} U^\dagger, \quad \Lambda_L = U \begin{pmatrix} e^{-\nu/2} & 0 \\ 0 & e^{\nu/2} \end{pmatrix} U^\dagger, \quad (8.7)$$

$$\text{with } U = (n_R, n_L) = \begin{pmatrix} e^{-i\varphi} \cos \frac{\theta}{2} & \sin \frac{\theta}{2} \\ \sin \frac{\theta}{2} & -e^{i\varphi} \cos \frac{\theta}{2} \end{pmatrix}, \quad e^{\pm\nu/2} = \sqrt{\frac{k_0 \pm |\vec{k}|}{m}},$$

with $n_{R,L}$ defined in Eqs. (7.98) and (7.99).

Using $U^\dagger n_R = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $U^\dagger n_L = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ we obtain:

$$\begin{aligned} \Lambda_R n_R &= e^{+\nu/2} n_R, & \Lambda_R n_L &= e^{-\nu/2} n_L, \\ \Lambda_L n_R &= e^{-\nu/2} n_R, & \Lambda_L n_L &= e^{+\nu/2} n_L. \end{aligned} \quad (8.8)$$

$\Rightarrow n \propto n_{R,L}$ is convenient choice for n in $u(k_r)$ of step 1.

Two independent standard solutions:

$$\begin{aligned} u_R(k) &= S(\Lambda) \sqrt{m} \begin{pmatrix} n_R \\ n_R \end{pmatrix} = \sqrt{m} \begin{pmatrix} \Lambda_R n_R \\ \Lambda_L n_R \end{pmatrix} = \begin{pmatrix} \sqrt{k_0 + |\vec{k}|} n_R \\ \sqrt{k_0 - |\vec{k}|} n_R \end{pmatrix}, \\ u_L(k) &= S(\Lambda) \sqrt{m} \begin{pmatrix} n_L \\ n_L \end{pmatrix} = \sqrt{m} \begin{pmatrix} \Lambda_R n_L \\ \Lambda_L n_L \end{pmatrix} = \begin{pmatrix} \sqrt{k_0 - |\vec{k}|} n_L \\ \sqrt{k_0 + |\vec{k}|} n_L \end{pmatrix}. \end{aligned} \quad (8.9)$$

Analogous procedure for ansatz $\psi(x) = e^{+ikx} v(k)$ leads to $(\not{k} + m)v = 0$ with the standard solutions:

$$v_R(k) = \begin{pmatrix} \sqrt{k_0 + |\vec{k}|} n_R \\ -\sqrt{k_0 - |\vec{k}|} n_R \end{pmatrix}, \quad v_L(k) = \begin{pmatrix} -\sqrt{k_0 - |\vec{k}|} n_L \\ \sqrt{k_0 + |\vec{k}|} n_L \end{pmatrix}. \quad (8.10)$$

Normalization: ($\bar{u} \equiv u^\dagger \gamma_0$)

$$\begin{aligned} \bar{u}_\sigma(k) u_\tau(k) &= 2m \delta_{\sigma\tau}, & \sigma, \tau &= R, L, \\ \bar{u}_\sigma(k) v_\tau(k) &= 0, \\ \bar{v}_\sigma(k) u_\tau(k) &= 0, \\ \bar{v}_\sigma(k) v_\tau(k) &= -2m \delta_{\sigma\tau}, \end{aligned} \quad (8.11)$$

Spin orientations of the solutions:

Spin operator in Dirac representation $D^{(\frac{1}{2},0)} \oplus D^{(0,\frac{1}{2})}$:

$$\vec{S} = \vec{J}_{(\frac{1}{2},0)} \oplus \vec{J}_{(0,\frac{1}{2})} = \frac{\vec{\sigma}}{2} \oplus \frac{\vec{\sigma}}{2} = \frac{1}{2} \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}. \quad (8.12)$$

Definition: *Helicity* = spin projection onto direction of flight \vec{e}

$$h \equiv \vec{e} \vec{S} = \frac{1}{2} \begin{pmatrix} \vec{e} \vec{\sigma} & 0 \\ 0 & \vec{e} \vec{\sigma} \end{pmatrix}. \quad (8.13)$$

Standard solutions $u_{R,L}$ and $v_{R,L}$ are helicity eigenstates:

$$h u_R = +\frac{1}{2} u_R, \quad h u_L = -\frac{1}{2} u_L, \quad h v_R = +\frac{1}{2} v_R, \quad h v_L = -\frac{1}{2} v_L. \quad (8.14)$$

\Rightarrow Particle solutions $\psi_{+,R/L}(x) = e^{-ikx} u_{R/L}(k)$ correspond to helicity states with $h = \pm\frac{1}{2}$.
 Note: Helicity content of antiparticle solutions $\psi_{-,R/L}(x) = e^{+ikx} v_{R/L}(k)$ clarified by QFT.

General solution of free (classical) Dirac equation:

$$\psi(x) = \int d\tilde{k} \sum_{\sigma=R,L} \left[\underbrace{a_{\sigma}(\vec{k})}_{\substack{\text{arbitrary functions of } \vec{k} \\ \hookrightarrow \text{creation/annihilation operators} \\ \text{for (anti)particles in QFT}}} u_{\sigma}(k) e^{-ikx} + \underbrace{b_{\sigma}^*(\vec{k})}_{\substack{\text{arbitrary functions of } \vec{k} \\ \hookrightarrow \text{creation/annihilation operators} \\ \text{for (anti)particles in QFT}}} v_{\sigma}(k) e^{+ikx} \right]. \quad (8.15)$$

8.2 Quantization of free Dirac fields

8.2.1 Quantization procedure

Classical Lagrangian:

Dirac equation obviously reproduced by

$$\mathcal{L} = \bar{\psi} (i\rlap{\not{\partial}} - m) \psi, \quad (8.16)$$

considering ψ and $\bar{\psi}$ as independent.

Euler–Lagrange equations:

$$0 = \frac{\partial \mathcal{L}}{\partial \bar{\psi}} - \partial^\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial^\mu \bar{\psi})} \right) = \frac{\partial \mathcal{L}}{\partial \bar{\psi}} = (i\rlap{\not{\partial}} - m) \psi, \quad (8.17)$$

$$0 = \frac{\partial \mathcal{L}}{\partial \psi} - \partial^\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial^\mu \psi)} \right) = -m\bar{\psi} - \partial^\mu (i\bar{\psi}\gamma_\mu) = -\bar{\psi} \left(m + i\overleftarrow{\not{\partial}} \right) \quad \text{adjoint Dirac eq.} \quad (8.18)$$

Canonical commutators:

Preliminary consideration:

Commutators $[a(\vec{k}), a^\dagger(\vec{p})]$, etc., lead to totally symmetric states $|\vec{k}, \vec{p}\rangle = a^\dagger(\vec{p})a^\dagger(\vec{k})|0\rangle$.

But: Fermions require totally antisymmetric states.

↔ Use ansatz with anticommutators $\{a(\vec{k}), a^\dagger(\vec{p})\}$ in quantization !

Canonical momentum variable to field $\psi(x)$:

$$\pi(x) = \frac{\partial \mathcal{L}}{\partial (\partial^0 \psi)} = i\bar{\psi}\gamma_0 = i\psi^\dagger. \quad (8.19)$$

Canonical equal-time anticommutators for quantization:

$$\begin{aligned} \left\{ \psi_\alpha(t, \vec{x}), i\psi_\beta^\dagger(t, \vec{y}) \right\} &= i \mathbf{1} \delta_{\alpha\beta} \delta(\vec{x} - \vec{y}), \quad \text{i.e.} \quad \left\{ \psi_\alpha(t, \vec{x}), \bar{\psi}_\beta(t, \vec{y}) \right\} = (\gamma_0)_{\alpha\beta} \delta(\vec{x} - \vec{y}), \\ \left\{ \psi_\alpha(t, \vec{x}), \psi_\beta(t, \vec{y}) \right\} &= \left\{ \bar{\psi}_\alpha(t, \vec{x}), \bar{\psi}_\beta(t, \vec{y}) \right\} = 0, \end{aligned} \quad (8.20)$$

Insertion of Fourier decompositions

$$\psi(x) = \int d\tilde{k} \sum_{\sigma=R,L} \left[a_\sigma(\vec{k}) u_\sigma(k) e^{-ikx} + b_\sigma^\dagger(\vec{k}) v_\sigma(k) e^{+ikx} \right], \quad (8.21)$$

$$\bar{\psi}(x) = \int d\tilde{k} \sum_{\sigma=R,L} \left[a_\sigma^\dagger(\vec{k}) \bar{u}_\sigma(k) e^{+ikx} + b_\sigma(\vec{k}) \bar{v}_\sigma(k) e^{-ikx} \right] \quad (8.22)$$

yields anticommutators for creation/annihilation operators:

$$\begin{aligned} \left\{ a_\sigma(\vec{k}), a_\tau^\dagger(\vec{p}) \right\} &= \left\{ b_\sigma(\vec{k}), b_\tau^\dagger(\vec{p}) \right\} = (2\pi)^3 2p_0 \delta(\vec{k} - \vec{p}) \delta_{\sigma\tau}, \\ \left\{ a_\sigma(\vec{k}), a_\tau(\vec{p}) \right\} &= 0, \quad \text{etc.} \end{aligned} \quad (8.23)$$

8.2.2 Particle states and Fock space

Fock space:

- Ground state $|0\rangle$ (*vacuum*, no particle excitation): $\langle 0| = (|0\rangle)^\dagger$, $\langle 0|0\rangle = 1$.

$$a_\sigma(\vec{p})|0\rangle = 0, \quad b_\sigma(\vec{p})|0\rangle = 0 \quad \forall \vec{p}. \quad (8.24)$$

- Excited states (particle states):

$$|f_\sigma(\vec{p}_1)\rangle = a_\sigma^\dagger(\vec{p}_1)|0\rangle \quad 1 \text{ fermion} \quad (8.25)$$

$$|\bar{f}_\sigma(\vec{p}_1)\rangle = b_\sigma^\dagger(\vec{p}_1)|0\rangle \quad 1 \text{ antifermion} \quad (8.26)$$

$$|f_\sigma(\vec{p}_1)f_\tau(\vec{p}_2)\rangle = a_\sigma^\dagger(\vec{p}_1)a_\tau^\dagger(\vec{p}_2)|0\rangle \quad 2 \text{ fermions} \quad (8.27)$$

$$= -a_\tau^\dagger(\vec{p}_2)a_\sigma^\dagger(\vec{p}_1)|0\rangle \quad (8.28)$$

$$= -|f_\tau(\vec{p}_2)f_\sigma(\vec{p}_1)\rangle \quad (8.29)$$

$$[= 0 \quad \text{if } \vec{p}_1 = \vec{p}_2 \text{ and } \sigma = \tau]$$

\vdots

Antisymmetric states \Rightarrow *Fermi-Dirac statistics*

- *Fock space* = Hilbert space spanned by all fermion and antifermion states:

$$\{ |0\rangle, |f_\sigma(\vec{p}_1)\rangle, |\bar{f}_\tau(\vec{p}_2)\rangle, |f_\sigma(\vec{p}_1)f_\tau(\vec{p}_2)\rangle, \dots \}$$

- (*Anti*)*Fermion number operators*:

$$\begin{aligned} N_{f_\sigma}(\vec{p}) &= a_\sigma^\dagger(\vec{p})a_\sigma(\vec{p}), & N_f &= \sum_{\sigma=R,L} \int d\tilde{p} N_{f_\sigma}(\vec{p}), \\ N_{\bar{f}_\sigma}(\vec{p}) &= b_\sigma^\dagger(\vec{p})b_\sigma(\vec{p}), & N_{\bar{f}} &= \sum_{\sigma=R,L} \int d\tilde{p} N_{\bar{f}_\sigma}(\vec{p}). \end{aligned} \quad (8.30)$$

\leftrightarrow Commutator relations:

$$\begin{aligned} [N_f, a_\sigma^\dagger(\vec{p})] &= +a_\sigma^\dagger(\vec{p}), & [N_f, a_\sigma(\vec{p})] &= -a_\sigma(\vec{p}), \\ [N_f, b_\sigma^\dagger(\vec{p})] &= 0, & [N_f, b_\sigma(\vec{p})] &= 0, \end{aligned} \quad (8.31)$$

analogously for $N_{\bar{f}}$ with $a \leftrightarrow b$.

Field operators and wave functions: (cf. scalar fields, Sect. 5.4)

One-particle wave function $\varphi(x)$ corresponding to fermion state $|f_\sigma(\vec{p})\rangle$:

$$\varphi_{f_\sigma(\vec{p})}(x) \equiv \langle 0|\psi(x)|f_\sigma(\vec{p})\rangle = \langle 0|\psi(x) a_\sigma^\dagger(\vec{p})|0\rangle = e^{-ipx} u_\sigma(p). \quad (8.32)$$

Space-time transformations of $|f\rangle$, $\psi(x)$, and $\varphi(x)$: $x \rightarrow x' = \Lambda x + a$

- Qm. states:

$$|f\rangle \rightarrow |f'\rangle = U(\Lambda, a)|f\rangle \quad \text{with } U = \text{unitary operator.} \quad (8.33)$$

\Leftrightarrow Transition amplitudes $\langle f'|g'\rangle = \langle f|U^\dagger U|g\rangle = \langle f|g\rangle = \text{invariant}$.

|| Comment:
 $U(\Lambda, a)$ are transformations in ∞ -dimensional representation of the Poincaré group, which is spanned by the particle states.

- Field operator:

$$\psi(x') = U(\Lambda, a) \underbrace{S(\Lambda)}_{\text{transformation of spin part of } \psi(x)} \psi(x) U^\dagger(\Lambda, a), \quad (8.34)$$

so that scalar products $\langle f|\dots\bar{\psi}_1(x)\dots\psi_2(x)\dots|g\rangle = \langle f'|\dots\bar{\psi}_1(x')\dots\psi_2(x')\dots|g'\rangle = \text{invariant}$.

- Wave function:

$$\begin{aligned} \varphi'(x') &= \langle 0|\psi(x')|f'\rangle = \langle 0| \underbrace{U(\Lambda, a)}_{= \langle 0| = \text{invariant}} S(\Lambda) \psi(x) \underbrace{U^\dagger(\Lambda, a)U(\Lambda, a)}_{= \mathbf{1}} |f\rangle \\ &= S(\Lambda) \langle 0|\psi(x)|f\rangle = S(\Lambda) \varphi(x). \end{aligned} \quad (8.35)$$

$$\Rightarrow \varphi'(x) = S(\Lambda) \varphi(\Lambda^{-1}(x - a)). \quad (8.36)$$

Wave functions transform like classical fields.

Properties of the particle states:

- Electric charge:

Electric current density: (= Noether current for symmetry $\psi \rightarrow \psi' = e^{-iq\omega}\psi$)

$$j^\mu = q\bar{\psi}\gamma^\mu\psi. \quad (8.37)$$

\Rightarrow Operator Q for conserved electric charge:

$$Q = q \int d^3x : \bar{\psi}\gamma^0\psi := q \int d\vec{p} \sum_{\sigma=R,L} [a_\sigma^\dagger(\vec{p})a_\sigma(\vec{p}) - b_\sigma^\dagger(\vec{p})b_\sigma(\vec{p})] = q(N_f - N_{\bar{f}}). \quad (8.38)$$

Charges of particle states:

$$\begin{aligned} Q |f_\sigma(\vec{p})\rangle &= q(N_f - N_{\bar{f}})a_\sigma^\dagger(\vec{p})|0\rangle \\ &= q \left(\underbrace{[N_f, a_\sigma^\dagger(\vec{p})]}_{=a_\sigma^\dagger(\vec{p})} - \underbrace{[N_{\bar{f}}, a_\sigma^\dagger(\vec{p})]}_{=0} \right) |0\rangle = q |f_\sigma(\vec{p})\rangle, \end{aligned} \quad (8.39)$$

$$Q |\bar{f}_\sigma(\vec{p})\rangle = \dots = -q |\bar{f}_\sigma(\vec{p})\rangle. \quad (8.40)$$

\Rightarrow Fermion f carries charge $+q$, antifermion \bar{f} charge $-q$.

- 4-momentum:

Energy-momentum tensor: (derived as for scalar fields)

$$\theta^{\mu\nu} = \frac{i}{2} \bar{\psi} \overleftrightarrow{\partial}^\nu \gamma^\mu \psi. \quad (8.41)$$

\Rightarrow Operator P^μ for 4-momentum:

$$P^\mu = \int d^3x : \theta^{0\mu} := \dots = \int d\vec{p} p^\mu \sum_\sigma [N_{f_\sigma}(\vec{p}) + N_{\bar{f}_\sigma}(\vec{p})]. \quad (8.42)$$

$$\Leftrightarrow [P^\mu, a_\sigma^\dagger(\vec{p})] = p^\mu a_\sigma^\dagger(\vec{p}), \quad [P^\mu, b_\sigma^\dagger(\vec{p})] = p^\mu b_\sigma^\dagger(\vec{p}). \quad (8.43)$$

4-momenta of the particle states:

$$P^\mu |f_\sigma(\vec{p})\rangle = [P^\mu, a_\sigma^\dagger(\vec{p})] |0\rangle = p^\mu a_\sigma^\dagger(\vec{p}) |0\rangle = p^\mu |f_\sigma(\vec{p})\rangle, \quad (8.44)$$

$$P^\mu |\bar{f}_\sigma(\vec{p})\rangle = \dots = p^\mu |\bar{f}_\sigma(\vec{p})\rangle. \quad (8.45)$$

\Rightarrow Both elementary fermion and antifermion states carry 4-momentum p^μ .

Alternative derivation of Eq. (8.43) via translation property of operator $\psi(x)$:

$$\underbrace{U(\Lambda = \mathbf{1}, a)^\dagger}_{=\exp\{-iP_\mu a^\mu\}} \psi(x) \underbrace{U(\mathbf{1}, a)}_{=\exp\{+iP_\mu a^\mu\}} = \psi(x - a). \quad (8.46)$$

\Leftrightarrow Taking a^μ infinitesimal yields

$$[P_\mu, \psi(x)] = -i\partial_\mu \psi(x). \quad (8.47)$$

\Leftrightarrow Commutators $[P^\mu, a_\sigma^{(\dagger)}(\vec{p})]$, etc. from plane-wave solution for $\psi(x)$.

- Spin and helicity:

Make use of Lorentz transformation property of ψ :

$$U(\Lambda, a = 0)^\dagger \psi(x) U(\Lambda, 0) = S(\Lambda) \psi(\Lambda^{-1}x), \quad (8.48)$$

where

- ◇ $U(\Lambda, 0) = \exp\{-\frac{i}{2}\omega_{\mu\nu}\mathcal{J}^{\mu\nu}\}$,
with $\mathcal{J}^{\mu\nu} =$ abstract operator of generalized total angular momentum,
- ◇ $S(\Lambda) = \begin{pmatrix} \Lambda_R & 0 \\ 0 & \Lambda_L \end{pmatrix} =$ spin transformation matrix in Dirac representation,
with generators $M^{\mu\nu}$.

$$\vec{S} = \frac{1}{2} \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix} = \text{spin part of } M^{\mu\nu}, \text{ see Eq. (8.12)}. \quad (8.49)$$

- ◇ $\psi(\Lambda^{-1}x) = \exp\{-\frac{i}{2}\omega_{\mu\nu}L^{\mu\nu}\} \psi(x)$,
where $L^{\mu\nu} = i(x^\mu\partial^\nu - x^\nu\partial^\mu) =$ generalized orbital angular momentum.

Connection between $\mathcal{J}^{\mu\nu}$, $M^{\mu\nu}$, and $L^{\mu\nu}$ derived upon taking $\omega_{\mu\nu}$ infinitesimal:

$$[\mathcal{J}^{\mu\nu}, \psi(x)] = -(M^{\mu\nu} + L^{\mu\nu}) \psi(x). \quad (8.50)$$

Restriction to rotational part of spin transformation:

$$[\vec{J}, \psi(x)] = -\vec{S} \psi(x), \quad \text{where } \vec{J} = \text{abstract generator for spin rotations.} \quad (8.51)$$

⇒ Helicities ($\vec{e} = \vec{p}/|\vec{p}|$ directions of \vec{J} , \vec{S}) of Fock states:

$$\begin{aligned} [\vec{e} \cdot \vec{J}, a_\sigma(\vec{p})] &= \left[\vec{e} \cdot \vec{J}, \int d^3x e^{ipx} u_\sigma^\dagger(p) \psi(x) \right] \\ &= \int d^3x e^{ipx} u_\sigma^\dagger(p) \underbrace{\vec{e} \cdot [\vec{J}, \psi(x)]}_{= -\vec{e} \cdot \vec{S} \psi(x) = -h \psi(x)} \\ &= - \int d^3x e^{ipx} \underbrace{u_\sigma^\dagger(p) h}_{= u_\sigma^\dagger(p) h^\dagger = [h u_\sigma(p)]^\dagger = \frac{1}{2} \text{sgn}(\sigma) u_\sigma(p)^\dagger,} \psi(x) \\ &\quad \text{where } \text{sgn}(R/L) = +/ -, \text{ see Eq. (8.14)} \\ &= -\frac{1}{2} \text{sgn}(\sigma) \int d^3x e^{ipx} u_\sigma^\dagger(p) \psi(x) \\ &= -\frac{1}{2} \text{sgn}(\sigma) a_\sigma(\vec{p}), \end{aligned} \quad (8.52)$$

$$\begin{aligned}
[\vec{e} \cdot \vec{J}, b_\sigma^\dagger(\vec{p})] &= \left[\vec{e} \cdot \vec{J}, \int d^3x e^{-ipx} v_\sigma^\dagger(p) \psi(x) \right] \\
&= \int d^3x e^{-ipx} v_\sigma^\dagger(p) \vec{e} \cdot [\vec{J}, \psi(x)] \\
&= - \int d^3x e^{-ipx} \underbrace{v_\sigma^\dagger(p) h}_{= [h v_\sigma(p)]^\dagger} \psi(x) \\
&= [h v_\sigma(p)]^\dagger = \frac{1}{2} \text{sgn}(\sigma) v_\sigma(p)^\dagger, \text{ see Eq. (8.14)} \\
&= -\frac{1}{2} \text{sgn}(\sigma) b_\sigma^\dagger(\vec{p}), \tag{8.53}
\end{aligned}$$

$$\hookrightarrow [\vec{e} \cdot \vec{J}, a_\sigma^\dagger(\vec{p})] = +\frac{1}{2} \text{sgn}(\sigma) a_\sigma^\dagger(\vec{p}), \quad [\vec{e} \cdot \vec{J}, b_\sigma(\vec{p})] = +\frac{1}{2} \text{sgn}(\sigma) b_\sigma(\vec{p}). \tag{8.54}$$

$$\begin{aligned}
\Rightarrow \vec{e} \cdot \vec{J} |f_\sigma(\vec{p})\rangle &= [\vec{e} \cdot \vec{J}, a_\sigma^\dagger(\vec{p})] |0\rangle = +\frac{1}{2} \text{sgn}(\sigma) a_\sigma^\dagger(\vec{p}) |0\rangle = +\frac{1}{2} \text{sgn}(\sigma) |f_\sigma(\vec{p})\rangle, \\
\vec{e} \cdot \vec{J} |\bar{f}_\sigma(\vec{p})\rangle &= \dots = -\frac{1}{2} \text{sgn}(\sigma) |\bar{f}_\sigma(\vec{p})\rangle, \tag{8.55}
\end{aligned}$$

i.e. fermion state $|f_{R/L}\rangle$ has helicity $\pm\frac{1}{2}$,
but antifermion state $|\bar{f}_{R/L}\rangle$ has helicity $\mp\frac{1}{2}$.

8.2.3 Fermion propagator

Definition:

$$\langle 0 | T \psi(x) \bar{\psi}(y) | 0 \rangle = i S_F(x, y) \quad x \overset{\bullet}{\longleftarrow} \overset{\bullet}{y} \quad \text{fermion propagator} \tag{8.56}$$

Note: Each (anti)commutation of two fermionic operators in $: \dots :$ and $T(\dots)$ products leads to a sign change !

Calculation of $S_F(x, y)$:

$$\begin{aligned}
\langle 0 | T \psi(x) \bar{\psi}(y) | 0 \rangle &= \langle 0 | T \int d\tilde{k} \sum_\sigma \left[a_\sigma(\vec{k}) u_\sigma(k) e^{-ikx} + b_\sigma^\dagger(\vec{k}) v_\sigma(k) e^{+ikx} \right] \\
&\quad \times \int d\tilde{p} \sum_\tau \left[a_\tau^\dagger(\vec{p}) \bar{u}_\tau(p) e^{+ipy} + b_\tau(\vec{p}) \bar{v}_\tau(p) e^{-ipy} \right] | 0 \rangle \\
&= \theta(x_0 - y_0) \langle 0 | \int d\tilde{k} \int d\tilde{p} \sum_{\sigma, \tau} e^{-i(kx - py)} u_\sigma(k) \bar{u}_\tau(p) a_\sigma(\vec{k}) a_\tau^\dagger(\vec{p}) | 0 \rangle \\
&\quad \hookrightarrow \text{fermion propagation from } y \text{ to } x \\
&\quad - \theta(y_0 - x_0) \langle 0 | \int d\tilde{k} \int d\tilde{p} \sum_{\sigma, \tau} e^{i(kx - py)} v_\sigma(k) \bar{v}_\tau(p) b_\tau(\vec{p}) b_\sigma^\dagger(\vec{k}) | 0 \rangle \\
&\quad \hookrightarrow \text{antifermion propagation from } x \text{ to } y \tag{8.57}
\end{aligned}$$

Using $\langle 0 | a_\sigma(\vec{k}) a_\tau^\dagger(\vec{p}) | 0 \rangle = (2\pi)^3 2k^0 \delta(\vec{k} - \vec{p}) \delta_{\sigma\tau}$, etc., this yields

$$\begin{aligned}
\langle 0 | T \psi(x) \bar{\psi}(y) | 0 \rangle &= \theta(x_0 - y_0) \int d\tilde{k} e^{-ik(x-y)} \underbrace{\sum_\sigma u_\sigma(k) \bar{u}_\sigma(k)}_{= \not{k} + m, \text{ completeness relation, see Exercise 10.1}} \\
&\quad - \theta(y_0 - x_0) \int d\tilde{k} e^{ik(x-y)} \underbrace{\sum_\sigma v_\sigma(k) \bar{v}_\sigma(k)}_{= \not{k} - m} \\
&= \theta(x_0 - y_0) \int d\tilde{k} e^{-ik(x-y)} (\not{k} + m) - \theta(y_0 - x_0) \int d\tilde{k} e^{ik(x-y)} (\not{k} - m) \\
&= \int \frac{d^4 k}{(2\pi)^4} e^{-ik(x-y)} \frac{i(\not{k} + m)}{k^2 - m^2 + i\epsilon}, \quad \text{as for scalar propagator, see Sect. 4.3.2} \\
&= (i\not{\partial}_x + m) \int \frac{d^4 k}{(2\pi)^4} \frac{ie^{-ik(x-y)}}{k^2 - m^2 + i\epsilon} = (i\not{\partial}_x + m) iD_F(x, y). \quad (8.58)
\end{aligned}$$

Consequences of $S_F(x, y) = (i\not{\partial}_x + m) D_F(x, y)$:

- $S_F(x, y)$ has the same causal properties as scalar propagator $D_F(x, y)$.
- Differential equation:

$$\begin{aligned}
(i\not{\partial}_x - m) S_F(x, y) &= (i\not{\partial}_x - m)(i\not{\partial}_x + m) D_F(x, y) = -(\square_x + m^2) D_F(x, y) \\
&= \delta(x - y). \quad (8.59)
\end{aligned}$$

$S_F(x, y)$ is inverse of the Dirac operator $(i\not{\partial} - m)$.

8.2.4 Connection between spins and statistics

Spin statistics theorem:

- Fields with integer spin $(0, 1, \dots)$ are quantized with commutators.
 \hookrightarrow States obey Bose–Einstein statistics.
- Fields with half-integer spin $(1/2, 3/2, \dots)$ are quantized with anticommutators.
 \hookrightarrow States obey Fermi–Dirac statistics.

“Proof”: otherwise several inconsistencies:

- Violation of causality, i.e. violation of

$$[\text{Obs}(x), \text{Obs}(y)] = 0 \quad \text{for } (x - y)^2 < 0. \quad (8.60)$$

- Energy spectrum not bounded from below. \rightarrow System unstable.
- Statement on relation between spin and BE / FD statistics supported by experiment.

Chapter 9

Interaction of scalar and fermion fields

9.1 Interacting fermion fields

Interaction Lagrangians with a Dirac fermion:

$$\mathcal{L} = \bar{\psi} (i\cancel{\partial} - m) \psi - V(\psi, \bar{\psi}, \Phi). \quad (9.1)$$

Properties of interaction potential V :

- Each term in V contains products of at least 3 fields (2 fields \rightarrow free propagation).
- $V =$ Lorentz invariant.
 $\hookrightarrow \psi$ always appears in products $\bar{\psi} \dots \psi$.
- V has mass dimension 4.
(Fields: $\dim[\phi] = \dim[A^\mu] = 1$, $\dim[\psi] = \frac{3}{2}$.)
- $V = V^\dagger =$ hermitian.

Examples:

- Yukawa interaction of a fermion and a scalar field ϕ :

$$V = y \phi \bar{\psi} \psi, \quad y = \text{dimensionless coupling strength.} \quad (9.2)$$

\hookrightarrow Basic interaction between fermions and the Higgs boson in the Standard Model.

- Electromagnetic interaction:

$$V = Q e \bar{\psi} \not{A} \psi, \quad e = \text{elementary charge, } Q = \text{relative fermion charge.} \quad (9.3)$$

Comments on the perturbative machinery:

↔ works as for scalar fields with few exceptions (signs!):

- EOMs: $\frac{\delta \mathcal{L}}{\delta \psi} = 0, \quad \frac{\delta \mathcal{L}}{\delta \bar{\psi}} = 0.$

↔ Operators $\frac{\partial}{\partial \psi}$, etc., anticommute with fermionic fields.

- Symmetries if fields Φ_k are bosonic or fermionic:

$$\Phi_k \rightarrow \Phi_k + \delta\omega_a \Delta_k^a(\Phi), \quad \delta \mathcal{L} = \delta\omega_a \partial_\mu K^{a,\mu}, \quad \delta\omega_a = \text{const.}, \quad (9.4)$$

Appropriate form of Noether currents (for signs):

$$j^{a,\mu} = \left\{ \Delta_k^a(\Phi) \frac{\partial}{\partial(\partial_\mu \Phi_k)} \right\} \mathcal{L} - K^{a,\mu}(\Phi), \quad \partial j^a = 0. \quad (9.5)$$

- Contractions for perturbative expansion of S -operator:

$$:\cdots \overbrace{\Phi_i \cdots \Phi_j} \cdots: \equiv (-1)^{P_{ij}} \langle 0 | T[\Phi_i \Phi_j] | 0 \rangle \cdot \cdots \Phi_{i-1} \Phi_{i+1} \cdots \Phi_{j-1} \Phi_{j+1} \cdots, \quad (9.6)$$

where P_{ij} = number of necessary commutations for reordering fermion operators.

⇒ With (9.6) Wick theorem also holds as usual.

Examples:

$$\begin{aligned} \diamond T[\psi_1 \bar{\psi}_2] &= : \psi_1 \bar{\psi}_2 : + \overbrace{\psi_1 \bar{\psi}_2} = : \psi_1 \bar{\psi}_2 : + \langle 0 | T[\psi_1 \bar{\psi}_2] | 0 \rangle, \\ \diamond T[\psi_1 \bar{\psi}_2 \psi_3 \bar{\psi}_4] &= : \psi_1 \bar{\psi}_2 \psi_3 \bar{\psi}_4 : \\ &+ : \psi_1 \bar{\psi}_2 : \langle 0 | T[\psi_3 \bar{\psi}_4] | 0 \rangle + : \psi_3 \bar{\psi}_4 : \langle 0 | T[\psi_1 \bar{\psi}_2] | 0 \rangle \\ &- : \psi_3 \bar{\psi}_2 : \langle 0 | T[\psi_1 \bar{\psi}_4] | 0 \rangle - : \psi_1 \bar{\psi}_4 : \langle 0 | T[\psi_3 \bar{\psi}_2] | 0 \rangle \\ &+ \langle 0 | T[\psi_1 \bar{\psi}_2] | 0 \rangle \langle 0 | T[\psi_3 \bar{\psi}_4] | 0 \rangle \\ &- \langle 0 | T[\psi_1 \bar{\psi}_4] | 0 \rangle \langle 0 | T[\psi_3 \bar{\psi}_2] | 0 \rangle. \end{aligned}$$

9.2 Yukawa theory

Definition of the model:

Lagrangian:

$$\begin{aligned}
 \mathcal{L}(\phi, \psi, \bar{\psi}) &= \mathcal{L}_{\psi,0} + \mathcal{L}_{\phi,0} + \mathcal{L}_{\text{int}}, & (9.7) \\
 \mathcal{L}_{\psi,0}(\psi, \bar{\psi}) &= : \bar{\psi} (i\not{\partial} - m_f) \psi :, & \text{Dirac fermion of mass } m_f, \\
 \mathcal{L}_{\phi,0}(\phi) &= : \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2}m_\phi^2 \phi^2 :, & \text{neutral scalar field of mass } m_\phi, \\
 \mathcal{L}_{\text{int}}(\phi, \psi, \bar{\psi}) &= -y : \phi \bar{\psi} \psi :, & \text{Yukawa interaction.}
 \end{aligned}$$

Hamiltonian:

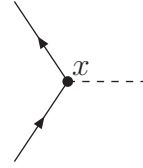
$$\mathcal{H}_{\text{int}}(\phi, \psi, \bar{\psi}) = -\mathcal{L}_{\text{int}}(\phi, \psi, \bar{\psi}), \quad \text{since no derivative involved.} \quad (9.8)$$

9.2.1 Feynman rules for the S-operator

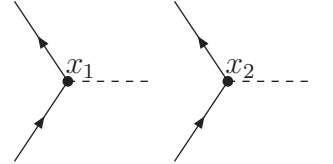
Expansion of the S-operator:

$$\begin{aligned}
 S &= T \exp \left[\int d^4x \, i\mathcal{L}_{\text{int}}(\phi(x), \psi(x), \bar{\psi}(x)) \right] \\
 &= T \exp \left[- \int d^4x \, iy : \phi(x) \bar{\psi}(x) \psi(x) : \right] \\
 &= \mathbf{1} + (-iy) T \left[\int d^4x : \phi(x) \bar{\psi}(x) \psi(x) : \right] \\
 &\quad + \frac{1}{2}(-iy)^2 T \left[\int d^4x_1 \int d^4x_2 : \phi(x_1) \bar{\psi}(x_1) \psi(x_1) : : \phi(x_2) \bar{\psi}(x_2) \psi(x_2) : \right] + \dots
 \end{aligned}$$

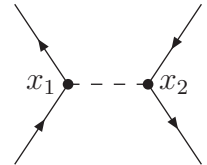
$$= \mathbf{1} + (-iy) \left[\int d^4x : \phi(x) \bar{\psi}(x) \psi(x) : \right]$$



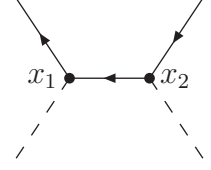
$$+ \frac{1}{2}(-iy)^2 \left[\int d^4x_1 \int d^4x_2 : \phi(x_1) \bar{\psi}(x_1) \psi(x_1) \phi(x_2) \bar{\psi}(x_2) \psi(x_2) : \right]$$



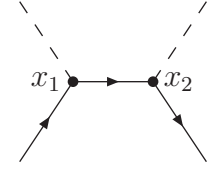
$$+ \int d^4x_1 \int d^4x_2 : \bar{\psi}(x_1) \psi(x_1) \overbrace{\phi(x_1) \phi(x_2)} \bar{\psi}(x_2) \psi(x_2) :$$



$$+ \int d^4x_1 \int d^4x_2 : \phi(x_1) \overline{\psi}(x_1) \overbrace{\psi(x_1) \overline{\psi}(x_2)} \psi(x_2) \phi(x_2) :$$



$$+ \int d^4x_1 \int d^4x_2 : \underbrace{\phi(x_2) \overline{\psi}(x_2) \psi(x_2) \overline{\psi}(x_1) \psi(x_1) \phi(x_1)}_{\text{convenient form for contractions with external states}} :$$



$$+ \int d^4x_1 \int d^4x_2 : \overline{\psi}(x_1) \psi(x_1) \overbrace{\psi(x_1) \overline{\psi}(x_2)} \psi(x_2) \phi(x_1) \phi(x_2) :$$



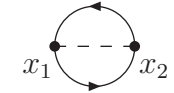
$$+ \int d^4x_1 \int d^4x_2 : \overline{\psi}(x_2) \psi(x_2) \overbrace{\psi(x_2) \overline{\psi}(x_1)} \psi(x_1) \phi(x_1) \phi(x_2) :$$



$$- \int d^4x_1 \int d^4x_2 : \phi(x_1) \phi(x_2) : \underbrace{\sum_{\alpha, \beta} \psi_\alpha(x_1) \overline{\psi}_\beta(x_2) \psi_\beta(x_2) \overline{\psi}_\alpha(x_1)}_{= \text{tr} [\psi(x_1) \overline{\psi}(x_2) \psi(x_2) \overline{\psi}(x_1)]}$$




$$- \int d^4x_1 \int d^4x_2 : \phi(x_1) \phi(x_2) \overbrace{\psi(x_1) \overline{\psi}(x_2) \psi(x_2) \overline{\psi}(x_1)} :]$$



+ ...

(9.9)

Feynman rules for graphical representation of the terms $\propto y^n$:

1. Draw all possible diagrams with n vertices  (any number of external lines, including disconnected diagrams).
2. Translate graphs into analytical expressions as follows:
 - External lines $\hat{=}$ non-contracted fields:

$$\begin{aligned}
 \phi(x) &= \text{---} \bullet_x \\
 \psi(x) &= \text{---} \bullet_x \\
 \bar{\psi}(x) &= \text{---} \bullet_x
 \end{aligned}
 \tag{9.10}$$

- Internal lines $\hat{=}$ contracted fields (=propagators):

$$\begin{aligned}
 \overbrace{\phi(x_1)\phi(x_2)} &= \bullet_{x_1} \text{---} \bullet_{x_2} \\
 \overbrace{\psi(x_1)\bar{\psi}(x_2)} &= \bullet_{x_1} \text{---} \bullet_{x_2}
 \end{aligned}
 \tag{9.11}$$

- Vertices $\hat{=}$ interaction terms:

$$-iy = \text{---} \bullet_x$$

(9.12)

3. Order terms opposite to the fermion flow indicated by the arrows.
4. For each closed fermion loop take Dirac trace and multiply by (-1) .
5. Integrate the sum of all terms according to

$$\frac{1}{n!} \int d^4x_1 \dots d^4x_n : \dots :$$
(9.13)

9.2.2 Feynman rules for S-matrix elements

Consider $n \rightarrow m$ particle process:

$$|i\rangle = a_{A_1}^\dagger \dots a_{A_n}^\dagger |0\rangle, \quad \langle f| = \langle 0| a_{B_m} \dots a_{B_1}$$
(9.14)

where $A_1, \dots, B_m =$ scalar or fermion fields $\phi, \psi, \bar{\psi}$.

\hookrightarrow Only contributions from terms $\propto a_{B_1}^\dagger \dots a_{B_m}^\dagger a_{A_n} \dots a_{A_1}$ in $S!$

1. Select terms graphically: $A_i, B_j \hat{=}$ external lines in diagrams

$$\begin{array}{c} \longrightarrow \bullet \\ x \end{array} \quad \text{incoming fermion/outgoing antifermion: } \psi(x) \text{ contains } a, b^\dagger \quad (9.15)$$

$$\begin{array}{c} \longleftarrow \bullet \\ x \end{array} \quad \text{outgoing fermion/incoming antifermion} \quad (9.16)$$

$$\begin{array}{c} - - - \bullet \\ x \end{array} \quad \text{incoming/outgoing real scalar} \quad (9.17)$$

2. Perform contractions of fields in normal-ordered products with external fields:

Typical manipulation:

$$\psi(x) a_\sigma^\dagger(\vec{p}) |0\rangle = \int d\vec{k} \sum_\tau \left[e^{-ikx} u_\tau(k) \underbrace{a_\tau(\vec{k}) a_\sigma^\dagger(\vec{p})}_{= -a_\sigma^\dagger(\vec{p}) a_\tau(\vec{k}) + \{a_\tau(\vec{k}), a_\sigma^\dagger(\vec{p})\}} + \dots \right] |0\rangle \quad (9.18)$$

$$\begin{aligned} &= -a_\sigma^\dagger(\vec{p}) a_\tau(\vec{k}) + \{a_\tau(\vec{k}), a_\sigma^\dagger(\vec{p})\} \\ &= (2\pi^3) 2p^0 \delta(\vec{p} - \vec{k}) \delta_{\sigma\tau} \\ &= e^{-ipx} u_\sigma(p) |0\rangle + \dots \end{aligned} \quad (9.19)$$

Define contractions with external fermion fields:

$$\begin{aligned} \overbrace{\psi(x) a_\sigma^\dagger(\vec{p})} &= e^{-ipx} u_\sigma(p) |0\rangle, \\ \overbrace{\bar{\psi}(x) b_\sigma^\dagger(\vec{p})} &= e^{ipx} \bar{v}_\sigma(p) |0\rangle, \\ \langle 0 | \overbrace{a_\sigma(\vec{p}) \bar{\psi}(x)} &= \langle 0 | e^{ipx} \bar{u}_\sigma(p), \\ \langle 0 | \overbrace{b_\sigma(\vec{p}) \psi(x)} &= \langle 0 | e^{-ipx} v_\sigma(p). \end{aligned} \quad (9.20)$$

3. Perform the integration of $\int d^4x_i$ after inserting the propagators:

$$iD_F(x_1, x_2) = \overbrace{\phi(x_1) \phi(x_2)} = \int \frac{d^4k}{(2\pi)^4} e^{-ik(x_1-x_2)} \frac{i}{k^2 - m_\phi^2 + i\epsilon} = \begin{array}{c} \bullet \text{---} \bullet \\ x_1 \quad x_2 \end{array} \quad (9.21)$$

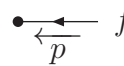
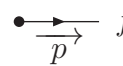
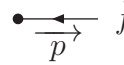
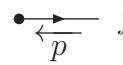
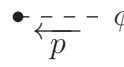
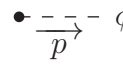
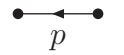
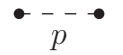
$$iS_F(x_1, x_2) = \overbrace{\psi(x_1) \bar{\psi}(x_2)} = \int \frac{d^4k}{(2\pi)^4} e^{-ik(x_1-x_2)} \frac{i}{\not{k} - m_f + i\epsilon} = \begin{array}{c} \bullet \longleftarrow \bullet \\ x_1 \quad x_2 \end{array} \quad (9.22)$$

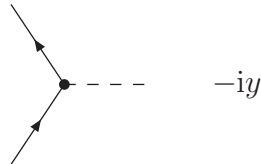
\leftrightarrow Momentum conservation at each vertex.

4. Calculate symmetry factor for each graph (are all 1 in this model).
5. Determine the sign for each graph from the permutation of fermionic operators.

Feynman rules for the transition matrix element \mathcal{M}_{fi} :

1. Determine all relevant Feynman diagrams:
 - $n \rightarrow m$ scattering process $\Rightarrow n + m$ external lines.
 - Order of perturbation theory \Rightarrow number of loops.
2. Impose momentum conservation at each vertex.
3. Insert the explicit expressions (fermionic terms ordered opposite to arrows):

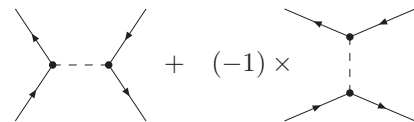
	$u_\sigma(p)$		$\bar{u}_\sigma(p)$
	$v_\sigma(p)$		$\bar{v}_\sigma(p)$
	1		1
	$\frac{i}{\not{p} - m_f + i\epsilon}$		$\frac{i}{p^2 - m_\phi^2 + i\epsilon}$



(9.23)

4. Integrate over all loop momenta p_l via $\int \frac{d^4 p_l}{(2\pi)^4}$.
5. For each closed fermion loop take Dirac trace and multiply by (-1) . Insert a relative sign between diagrams that result from interchanging external fermion lines.

Example:



6. The coherent sum of all diagrams yields $i\mathcal{M}_{fi}$.

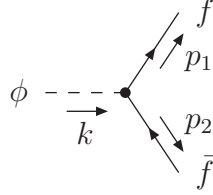
Example: the decay $\phi \rightarrow f\bar{f}$ in lowest order

\hookrightarrow Practically identical to the decays $H \rightarrow f\bar{f}$ ($f = b, \tau$, etc.) of the Standard Model Higgs boson.

Process:

$$\phi(k) \rightarrow f_\sigma(p_1) + \bar{f}_\tau(p_2), \quad \sigma, \tau = \text{helicities.} \quad (9.24)$$

Lowest-order diagram:



Amplitude:

$$\begin{aligned} i\mathcal{M} &= -iy \bar{u}_\sigma(p_1) v_\tau(p_2). & (9.25) \\ \Rightarrow \sum_{\text{pol}} |\mathcal{M}|^2 &= y^2 \sum_{\sigma, \tau} (\bar{u}_\sigma(p_1) v_\tau(p_2)) \underbrace{(\bar{u}_\sigma(p_1) v_\tau(p_2))^*}_{= (\bar{u}_\sigma(p_1) v_\tau(p_2))^\dagger} \\ &= \bar{v}_\tau(p_2) u_\sigma(p_1) \\ &= y^2 \sum_{\alpha, \beta} \underbrace{\left(\sum_\sigma u_\sigma(p_1)_\alpha \bar{u}_\sigma(p_1)_\beta \right)}_{= (\not{p}_1 + m_f)_{\alpha\beta}} \underbrace{\left(\sum_\tau v_\tau(p_2)_\beta \bar{v}_\tau(p_2)_\alpha \right)}_{= (\not{p}_2 - m_f)_{\beta\alpha}} \\ &= y^2 \text{Tr} \{ (\not{p}_1 + m_f) (\not{p}_2 - m_f) \} \\ &= 4y^2 (p_1 p_2 - m_f^2), \quad m_\phi^2 = k^2 = (p_1 + p_2)^2 = 2m_f^2 + 2p_1 p_2 \\ &= 2y^2 (m_\phi^2 - 4m_f^2). & (9.26) \end{aligned}$$

Partial decay width:

$$\begin{aligned} \Gamma_{\phi \rightarrow f\bar{f}} &= \frac{1}{2m_\phi} \int d\Phi_2 \sum_{\text{pol}} |\mathcal{M}|^2, \quad \text{for phase space } \Phi_2, \text{ see Exercise 7.1} \\ &= \frac{1}{2m_\phi} \frac{1}{(2\pi)^2} \frac{\sqrt{m_\phi^4 - 4m_\phi^2 m_f^2}}{8m_\phi^2} \underbrace{\int d\Omega_1}_{= 4\pi} 2y^2 (m_\phi^2 - 4m_f^2) \\ &= \frac{y^2 m_\phi}{8\pi} \left(1 - \frac{4m_f^2}{m_\phi^2} \right)^{\frac{3}{2}}. & (9.27) \end{aligned}$$

Part III

Quantization of vector-boson fields

Chapter 10

Free vector-boson fields

10.1 Classical Maxwell equations

Electromagnetic fields:

- *4-vector potential:* $A^\mu = (\Phi, \vec{A})$, $\Phi =$ scalar potential (\neq Lorentz scalar),
 $\vec{A} =$ 3-vector potential.

- *Field-strength tensor:*

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu, \quad (10.1)$$

in components: $\vec{E} = -\vec{\nabla}\Phi - \dot{\vec{A}}$, $\vec{B} = \vec{\nabla} \times \vec{A}$.

- *Elmg. gauge invariance:* Field strengths do not change under

$$A^\mu \rightarrow A'^\mu = A^\mu + \partial^\mu \omega, \quad \omega = \omega(x) = \text{arbitrary function of } x. \quad (10.2)$$

A^μ is not uniquely fixed by $F^{\mu\nu}$ (i.e. \vec{E}, \vec{B}), and a specific choice is called a *gauge*.

Examples:

- *Covariant (Lorenz) gauge:* $\partial A = 0$.
 $\hookrightarrow A^\mu$ unique up to gauge transformations with $\square\omega = 0$.
- *Radiation gauge:* $A^0 = \Phi = 0$, $\vec{\nabla} \cdot \vec{A} = 0$.

Maxwell equations:

$$0 = \underbrace{j^\nu}_{= \text{external current density} = 0 \text{ for free } A^\mu \text{ fields}} = \partial_\mu F^{\mu\nu} = \square A^\nu - \partial^\nu(\partial A). \quad (10.3)$$

Lagrangian density:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = \text{gauge invariant} \quad (10.4)$$

$$= -\frac{1}{2}(\partial_\mu A_\nu)(\partial^\mu A^\nu) + \frac{1}{2}(\partial_\mu A_\nu)(\partial^\nu A^\mu) \stackrel{\text{part. int.}}{=} \frac{1}{2}A_\mu(g^{\mu\nu}\square - \partial^\mu\partial^\nu)A_\nu. \quad (10.5)$$

Check EOM:

$$\begin{aligned} 0 &= \frac{\partial\mathcal{L}}{\partial A_\rho} - \partial_\sigma \left(\frac{\partial\mathcal{L}}{\partial(\partial_\sigma A_\rho)} \right) = -\partial_\sigma \left[-\frac{1}{2}\delta_\mu^\sigma\delta_\nu^\rho(\partial^\mu A^\nu) \cdot 2 + \frac{1}{2}\delta_\mu^\sigma\delta_\nu^\rho(\partial^\nu A^\mu) \cdot 2 \right] \\ &= -\partial_\sigma(-\partial^\sigma A^\rho + \partial^\rho A^\sigma) = \partial_\sigma F^{\sigma\rho}. \end{aligned}$$

Solution via plane elmg. waves:

Ansatz: $A^\mu(x) = \underbrace{\varepsilon^\mu(k)}_{\text{constant polarization vector}} e^{-ikx}.$

Convenient choice: covariant gauge

- EOM: $\square A^\mu = 0 \Rightarrow k^2 = 0$, i.e. k^μ light-like.
- Gauge: $\partial A = 0 \Rightarrow k^\mu \varepsilon(k) = 0.$
 $\hookrightarrow \varepsilon^\mu(k)$ is spanned by 3 independent vectors.

Convenient: *helicity basis*

Example: $k^\mu = k_0(1, 0, 0, 1), \quad \varepsilon_\pm^\mu(k) \equiv (0, 1, \pm i, 0)/\sqrt{2} = [\varepsilon_\mp^\mu(k)]^*.$

Basis: $\underbrace{\varepsilon_\pm^\mu(k)}_{\substack{2 \text{ physical} \\ \text{polarizations}}}, \quad \underbrace{k^\mu}_{\substack{\text{"unphysical polarization",} \\ \text{part of the gauge degree of} \\ \text{freedom}}}$

Normalization: $\varepsilon_\lambda^\mu(k) \varepsilon_{\lambda',\mu}(k)^* = -\delta_{\lambda\lambda'}.$

\Rightarrow General solution of Maxwell eq.:

$$A^\mu(x) = \int d\tilde{k} \sum_{\lambda=\pm} \left[\underbrace{a_\lambda(\vec{k})}_{\substack{\text{arbitrary functions,} \\ \text{will become creation/annihilation} \\ \text{operators in QFT}}} \varepsilon_\lambda^\mu(k) e^{-ikx} + \underbrace{a_\lambda^*(\vec{k})}_{\substack{\text{arbitrary functions,} \\ \text{will become creation/annihilation} \\ \text{operators in QFT}}} \varepsilon_\lambda^\mu(k)^* e^{+ikx} \right] = A^\mu(x)^*. \quad (10.6)$$

10.2 Proca equation

Aim: description of massive spin-1 particles

\Leftrightarrow free field modes with momentum k^μ should obey $k^2 = m^2 \neq 0$.

\Rightarrow Generalization of Maxwell eq.:

$$(\square + m^2)V^\mu - \partial^\mu(\partial V) = 0, \quad \text{Proca equation.} \quad (10.7)$$

Features of the free Proca field V^μ :

- Transversality:

$$\partial_\mu(\dots) \Rightarrow m^2(\partial V) = 0, \quad \text{i.e. } \partial V = 0 \text{ automatically fulfilled.}$$

$$\Rightarrow \text{Eq. (10.7)} \Leftrightarrow (\square + m^2)V^\mu = 0 \text{ and } \partial V = 0.$$

- Lagrangian: (real V^μ)

$$\mathcal{L} = -\frac{1}{4}V_{\mu\nu}V^{\mu\nu} + \frac{1}{2}m^2V_\mu V^\mu, \quad V^{\mu\nu} = \partial^\mu V^\nu - \partial^\nu V^\mu. \quad (10.8)$$

\Leftrightarrow Proca eq. as EOM.

Note: $\mathcal{L} \neq$ invariant under $V^\mu \rightarrow V'^\mu = V^\mu + \partial^\mu \omega$.

- Solution via plane waves:

$$V^\mu(x) = \int d\vec{k} \sum_{\lambda=0,\pm} \left[a_\lambda(\vec{k}) \varepsilon_\lambda^\mu(k) e^{-ikx} + a_\lambda^*(\vec{k}) \varepsilon_\lambda^\mu(k)^* e^{+ikx} \right]. \quad (10.9)$$

Note: 3 physical polarization states exist for massive spin-1 fields for each k^μ .

Helicity basis for $k^\mu = (k_0, 0, 0, |\vec{k}|)$, $k_0 = \sqrt{\vec{k}^2 + m^2}$:

$$\begin{aligned} \lambda = \pm 1: & \quad \varepsilon_\pm^\mu(k) = (0, 1, \pm i, 0)/\sqrt{2}, & \text{transverse polarizations,} \\ \lambda = 0: & \quad \varepsilon_0^\mu(k) = (|\vec{k}|, 0, 0, -k_0)/m, & \text{longitudinal polarization.} \end{aligned} \quad (10.10)$$

- Complex Proca field:

$$\mathcal{L} = -\frac{1}{2}V_{\mu\nu}^\dagger V^{\mu\nu} + m^2 V_\mu^\dagger V^\mu. \quad (10.11)$$

\Leftrightarrow V_μ and V_μ^\dagger obey the Proca eq.,

$a_\lambda(\vec{k})$ and $a_\lambda^*(\vec{k})$ become independent amplitudes in Eq. (10.9).

10.3 Quantization of the elmg. field

10.3.1 Preliminaries

$A_\mu = A_\mu^\dagger$ hermitian field operator with

$$\mathcal{L} = -\frac{1}{4} : F_{\mu\nu} F^{\mu\nu} : , \quad F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu. \quad (10.12)$$

\hookrightarrow Canonical momentum variable: $\pi^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_0 A_\mu)} = F^{\mu 0}$, i.e. $\pi^0 = 0$.

\Rightarrow Contradiction to canonical commutator $[A_0(t, \vec{x}), \pi_0(t, \vec{y})] \neq 0$!

Origin of the problem:

A^μ contains unphysical degrees of freedom because of gauge invariance.

Possible solutions:

- Fix the gauge of the operator A^μ by constraints.
 \hookrightarrow No quantization of unphysical degrees of freedom.

Example: Radiation gauge $A^0 = \Phi = 0$, $\vec{\nabla} \cdot \vec{A} = 0$.

Disadvantage:

Covariant gauge $\partial A = 0$ not compatible with $[A_\mu(t, \vec{x}), \pi_\nu(t, \vec{y})] = i g_{\mu\nu} \delta(\vec{x} - \vec{y})$.

- Impose gauge constraints on physical states (instead of field operators).
 \hookrightarrow Unphysical degrees of freedom are quantized, but can be decoupled in observables.

Corresponding *Gupta–Bleuler* procedure described in the following.

10.3.2 Gupta–Bleuler quantization:

Starting assumptions:

1. Add *covariant gauge-fixing term* to Lagrangian:

$$\mathcal{L} = -\frac{1}{4} : F_{\mu\nu} F^{\mu\nu} : - \underbrace{\frac{1}{2\xi} : (\partial A)^2 :}_{\text{gauge-fixing term with arbitrary gauge parameter } \xi > 0} \quad (10.13)$$

2. Demand constraint not as operator equation $\partial A = 0$, but as constraint on expectation values:

$$\langle \psi | \partial A | \psi \rangle \stackrel{!}{=} 0, \quad \forall |\psi\rangle \in \mathcal{H}_{\text{phys}} = \text{Hilbert space of physical states.} \quad (10.14)$$

Quantization:

- Canonical momentum variable:

$$\pi^\mu = F^{\mu 0} - \frac{1}{\xi} g^{\mu 0} (\partial A). \quad (10.15)$$

- Canonical commutators:

$$\begin{aligned} [A^\mu(t, \vec{x}), \pi^\nu(t, \vec{y})] &= i g^{\mu\nu} \delta(\vec{x} - \vec{y}), \\ [A^\mu(t, \vec{x}), A^\nu(t, \vec{y})] &= 0, \\ [\pi^\mu(t, \vec{x}), \pi^\nu(t, \vec{y})] &= 0. \end{aligned} \quad (10.16)$$

⇒ For $\xi = 1$ (used in the following!):

$$\begin{aligned} [A^\mu(t, \vec{x}), \dot{A}^\nu(t, \vec{y})] &= -i [g^{\mu\nu} + g^{\mu 0} g^{\nu 0} (\xi - 1)] \delta(\vec{x} - \vec{y}) \stackrel{\xi=1}{=} -i g^{\mu\nu} \delta(\vec{x} - \vec{y}), \\ [A^\mu(t, \vec{x}), A^\nu(t, \vec{y})] &= 0, \\ [\dot{A}^\mu(t, \vec{x}), \dot{A}^\nu(t, \vec{y})] &= -i(\xi - 1) [g^{\mu 0} g^{\nu k} + g^{\mu k} g^{\nu 0}] \partial_{k,x} \delta(\vec{x} - \vec{y}) \stackrel{\xi=1}{=} 0, \end{aligned} \quad (10.17)$$

i.e. A^k ($k = 1, 2, 3$) behave like scalar fields, but A^0 has “wrong” sign in commutator.

- EOM:

$$\square A_\mu - \left(1 - \frac{1}{\xi}\right) \partial_\mu (\partial A) \stackrel{\xi=1}{=} \square A_\mu = 0. \quad (10.18)$$

Problem: Find $A^\mu(x)$ obeying Eqs. (10.14), (10.17), and (10.18).

↔ Fourier ansatz:

$$A^\mu(x) = \int d\tilde{k} \sum_{\lambda=0}^3 \left[a_\lambda(\vec{k}) \varepsilon_\lambda^\mu(k) e^{-ikx} + a_\lambda^\dagger(\vec{k}) \varepsilon_\lambda^\mu(k)^* e^{+ikx} \right] = A^\mu(x)^\dagger, \quad (10.19)$$

with the “extended helicity basis” for $k^\mu = k_0(1, \vec{e})$:

$$\begin{aligned} \varepsilon_{1,2}^\mu(k) &= (0, \vec{e}_{1,2}), \quad \text{transversal d.o.f.} \\ &\text{where } \vec{e}_{1,2}^2 = 1, \quad \vec{e} \cdot \vec{e}_{1,2} = 0, \quad \vec{e}_1 \cdot \vec{e}_2 = 0, \quad \vec{e}_1 \times \vec{e}_2 = \vec{e}, \\ &\text{so that } \varepsilon_\pm^\mu(k) = \left(\varepsilon_1^\mu(k) \pm i \varepsilon_2^\mu(k) \right) / \sqrt{2} = \text{helicity states,} \\ \varepsilon_3^\mu(k) &= (0, \vec{e}), \quad \text{longitudinal d.o.f.} \\ \varepsilon_0^\mu(k) &= (1, \vec{0}), \quad \text{scalar d.o.f.} \end{aligned} \quad (10.20)$$

$$\Rightarrow \varepsilon_\lambda(k) \cdot \varepsilon_{\lambda'}(k)^* = g_{\lambda\lambda'}, \quad \sum_{\lambda, \lambda'=0}^3 (-g_{\lambda\lambda'}) \varepsilon_\lambda^\mu(k) \varepsilon_{\lambda'}^\nu(k)^* = -g^{\mu\nu}. \quad (10.21)$$

Insertion of ansatz:

- EOM (10.18) only demands $k^2 = 0$, i.e. $k_0 = |\vec{k}|$.
- Commutators (10.17) lead to

$$\begin{aligned} [a_\lambda(\vec{k}), a_{\lambda'}^\dagger(\vec{k}')] &= -g_{\lambda\lambda'} 2k_0 (2\pi)^3 \delta(\vec{k} - \vec{k}'), \\ [a_\lambda(\vec{k}), a_{\lambda'}(\vec{k}')] &= [a_\lambda^\dagger(\vec{k}), a_{\lambda'}^\dagger(\vec{k}')] = 0. \end{aligned} \quad (10.22)$$

$\hookrightarrow a_{1,2,3}^{(\dagger)}$ as for scalar field, but the roles of a_0 and a_0^\dagger are interchanged.

- Gauge condition (10.14), $\langle \psi | \partial A | \psi \rangle = 0$, already results from demand $\underbrace{\partial A^{(+)} | \psi \rangle}_{\text{only } e^{-ikx} \text{ part}} \stackrel{!}{=} 0$:

$$0 = (\partial A^{(+)} | \psi \rangle)^\dagger = \langle \psi | \partial A^{(-)}, \quad \langle \psi | \partial A | \psi \rangle = \langle \psi | \partial A^{(+)} + \partial A^{(-)} | \psi \rangle = 0.$$

\hookrightarrow Condition on a_λ :

$$\begin{aligned} 0 &= \partial A^{(+)} | \psi \rangle = \int d\vec{k} e^{-ikx} \sum_{\lambda=0}^3 \underbrace{(k \cdot \varepsilon_\lambda(k))}_{\substack{k \cdot \varepsilon_0 = k_0 = -k \cdot \varepsilon_3, \\ k \cdot \varepsilon_{1,2} = 0}} a_\lambda(\vec{k}) | \psi \rangle. \\ \Rightarrow 0 &= [a_0(\vec{k}) - a_3(\vec{k})] | \psi \rangle. \end{aligned} \quad (10.23)$$

Change of basis:

$$\begin{aligned} \varepsilon_L^\mu(k) &\equiv (1, \vec{e}) = \varepsilon_0^\mu(k) + \varepsilon_3^\mu(k), & \varepsilon_N^\mu(k) &\equiv (1, -\vec{e}) = \varepsilon_0^\mu(k) - \varepsilon_3^\mu(k), \\ \text{i.e. } \varepsilon_0 &= \frac{1}{2}(\varepsilon_L + \varepsilon_N), & \varepsilon_3 &= \frac{1}{2}(\varepsilon_L - \varepsilon_N), & \varepsilon_L^2 &= \varepsilon_N^2 = 0, & \varepsilon_L \cdot \varepsilon_N &= 2. \end{aligned}$$

$$\Rightarrow a_0(\vec{k}) \varepsilon_0^\mu(k) + a_3(\vec{k}) \varepsilon_3^\mu(k) = a_L(\vec{k}) \varepsilon_L^\mu(k) + a_N(\vec{k}) \varepsilon_N^\mu(k),$$

$$\text{with } a_L = \frac{1}{2}(a_0 + a_3), \quad a_N = \frac{1}{2}(a_0 - a_3),$$

$$[a_L(\vec{k}), a_L^\dagger(\vec{k}')] = [a_N(\vec{k}), a_N^\dagger(\vec{k}')] = 0,$$

$$[a_L(\vec{k}), a_N^\dagger(\vec{k}')] = [a_N(\vec{k}), a_L^\dagger(\vec{k}')] = -k^0 (2\pi)^3 \delta(\vec{k} - \vec{k}') \quad \text{etc.} \quad (10.24)$$

\Rightarrow Gauge condition (10.23):

$$a_N(\vec{k}) | \psi \rangle = 0. \quad (10.25)$$

Fock space:

- 1-particle states:

$$|\vec{k}, \lambda\rangle \equiv a_{\lambda}^{\dagger}(\vec{k}) |0\rangle, \quad \text{where } a_{\lambda}(\vec{k}) |0\rangle = 0, \quad \lambda = 0, 1, 2, 3. \quad (10.26)$$

Wave packets:

$$|f_{\lambda}\rangle = \int d\tilde{k} f_{\lambda}(\vec{k}) |\vec{k}, \lambda\rangle. \quad (10.27)$$

Orthogonality / normalization of states:

$$\begin{aligned} \langle f_{\lambda} | f_{\lambda'} \rangle &= \int d\tilde{k} \int d\tilde{k}' f_{\lambda}^*(\vec{k}) f_{\lambda'}(\vec{k}') \underbrace{\langle 0 | a_{\lambda}(\vec{k}) a_{\lambda'}^{\dagger}(\vec{k}') | 0 \rangle}_{=-g_{\lambda\lambda'} 2k_0 (2\pi)^3 \delta(\vec{k}-\vec{k}')} \\ &= -g_{\lambda\lambda'} \int d\tilde{k} f_{\lambda}^*(\vec{k}) f_{\lambda'}(\vec{k}), \quad (\text{no summation } \Sigma_{\lambda}) \end{aligned} \quad (10.28)$$

$$\hookrightarrow \|f_0\|^2 = \langle f_0 | f_0 \rangle = - \int d\tilde{k} |f_0(\vec{k})|^2 < 0, \quad \text{indefinite metric !}$$

$$\|f_{\lambda}\|^2 > 0, \quad \lambda = 1, 2, 3. \quad (10.29)$$

- Classification of states:

$$\begin{aligned} \mathcal{H} &\equiv \left\{ |\psi\rangle : \partial A^{(+)} |\psi\rangle = 0 \right\}, \\ \mathcal{H}_T &\equiv \left\{ |\psi_T\rangle : |\psi_T\rangle \text{ generated by } a_{1,2}^{\dagger}(\vec{k}) \right\}, \\ \mathcal{H}' &\equiv \left\{ |\psi'\rangle : |\psi'\rangle \text{ generated by } a_{0,3}^{\dagger}(\vec{k}) \text{ and } \partial A^{(+)} |\psi'\rangle = 0 \right\}. \end{aligned} \quad (10.30)$$

Properties:

- $\mathcal{H}' \subset \mathcal{H}$, trivial.
- $\mathcal{H}_T \subset \mathcal{H}$, since $\partial A^{(+)} |\psi_T\rangle = 0 \forall |\psi_T\rangle \in \mathcal{H}_T$.
- $\mathcal{H} = \mathcal{H}_T \otimes \mathcal{H}'$, completeness.

Note: $|\psi\rangle = |\psi_T\rangle \otimes |\psi'\rangle$ without ordering issues, since $[a_{\lambda}^{(\dagger)}, a_{\lambda'}^{(\dagger)}] = 0$ for $\lambda \neq \lambda'$.

Inspection of $|\psi'\rangle$:

$|\psi'\rangle =$ linear combination of

$$a_{\lambda_n}^{\dagger} \dots a_{\lambda_1}^{\dagger} |0\rangle, \quad \lambda_i = 0, 3, \quad \text{or equivalently } \lambda_i = L, N. \quad (10.31)$$

Gauge condition (10.25):

$$\begin{aligned} 0 &\stackrel{!}{=} a_N(\vec{p}) a_N^{\dagger}(\vec{k}_1) \dots a_N^{\dagger}(\vec{k}_{n_N}) a_L^{\dagger}(\vec{k}_1) \dots a_L^{\dagger}(\vec{k}_{n_L}) |0\rangle \\ &= a_N^{\dagger}(\vec{k}_1) \dots a_N^{\dagger}(\vec{k}_{n_N}) \underbrace{a_N(\vec{p}) a_L^{\dagger}(\vec{k}_1) \dots a_L^{\dagger}(\vec{k}_{n_L}) |0\rangle}_{\neq 0 \text{ for } n_L > 0, \text{ see (10.24)}}. \end{aligned} \quad (10.32)$$

$\Rightarrow |\psi'\rangle$ only generated by $a_N^{\dagger}(\vec{k})$!

General form of $|\psi'\rangle$:

$$\begin{aligned}
|\psi'\rangle &= c_0 |0\rangle + |N, 1\rangle + |N, 2\rangle + \dots, \\
|N, n\rangle &= \text{generated from } n > 0 \text{ ops. } a_N^\dagger. \\
\langle N, n | N, n'\rangle &= 0, \quad \text{since } [a_N, a_N^\dagger] = 0. \\
\Rightarrow \|\psi'\|^2 &= \langle \psi' | \psi'\rangle = |c_0|^2 \langle 0 | 0\rangle = |c_0|^2.
\end{aligned} \tag{10.33}$$

Physical observables:

General form:

$$\begin{aligned}
G &= \int d\tilde{k} g(\vec{k}) \sum_{\lambda=0}^3 a_\lambda^\dagger(\vec{k}) a_\lambda(\vec{k}) \eta_\lambda, \quad \eta_\lambda = \begin{cases} -1, & \lambda = 0, \\ +1, & \lambda = 1, 2, 3. \end{cases} \\
&= \underbrace{\int d\tilde{k} g(\vec{k}) \sum_{\lambda=1,2} a_\lambda^\dagger(\vec{k}) a_\lambda(\vec{k})}_{\equiv G_T} + \underbrace{\int d\tilde{k} g(\vec{k}) \sum_{\lambda=0,3} a_\lambda^\dagger(\vec{k}) a_\lambda(\vec{k})}_{\equiv G'}.
\end{aligned} \tag{10.34}$$

Expectation value in state $|\psi\rangle = |\psi_T\rangle \otimes |\psi'\rangle$:

$$\begin{aligned}
\langle \psi | G | \psi \rangle &= \langle \psi_T | G_T | \psi_T \rangle \underbrace{\langle \psi' | \psi' \rangle}_{= |c_0|^2} + \langle \psi_T | \psi_T \rangle \underbrace{\langle \psi' | G' | \psi' \rangle}, \\
&= \int d\tilde{k} g(\vec{k}) \langle \psi' | a_3^\dagger a_3 - a_0^\dagger a_0 | \psi' \rangle \\
&= -2 \int d\tilde{k} g(\vec{k}) \langle \psi' | a_N^\dagger a_L + a_L^\dagger a_N | \psi' \rangle = 0 \\
\frac{\langle \psi | G | \psi \rangle}{\langle \psi | \psi \rangle} &= \frac{\langle \psi_T | G_T | \psi_T \rangle |c_0|^2}{\langle \psi_T | \psi_T \rangle |c_0|^2} = \frac{\langle \psi_T | G_T | \psi_T \rangle}{\langle \psi_T | \psi_T \rangle}.
\end{aligned} \tag{10.35}$$

$\Rightarrow |\psi'\rangle$ part of $|\psi\rangle$ irrelevant for observables.

Hilbert space of physical states: $\mathcal{H}_{\text{phys}} = \mathcal{H}/\mathcal{H}' \sim \mathcal{H}_T$.

Example: 4-momentum

$$P^\mu = \int d\tilde{k} k^\mu \underbrace{\sum_{\lambda=0}^3 \eta_\lambda}_{\text{can be replaced by } \sum_{\lambda=1,2} \text{ in } \mathcal{H}_T} a_\lambda^\dagger(\vec{k}) a_\lambda(\vec{k}). \tag{10.36}$$

Comment:

$\mathcal{H}_{\text{phys}} \sim \mathcal{H}_T$ holds for general $\xi \neq 1$ as well, but proof non-trivial.

10.4 Photon propagator

Definition:

$$iD_F^{\mu\nu}(x, y) = \langle 0|TA^\mu(x)A^\nu(y)|0\rangle, \quad \begin{array}{c} \mu \\ \bullet \\ x \end{array} \text{---} \begin{array}{c} \nu \\ \bullet \\ y \end{array} \quad \text{photon propagator.} \quad (10.37)$$

Explicit calculation:

- Insert Fourier representation (10.19) (for $\xi = 1$) and use canonical commutators (10.22), or
- derive and solve differential equation for propagator (shown for general ξ in the following).

Wave operator and EOM:

$$\mathcal{D}^{\mu\nu} = g^{\mu\nu}\square - \left(1 - \frac{1}{\xi}\right)\partial^\mu\partial^\nu, \quad \text{EOM: } \mathcal{D}^{\mu\nu}A_\nu(x) = 0 \quad (10.38)$$

$$\begin{aligned} \mathcal{D}^{\mu\nu} &= \underbrace{\left[g^{\mu\nu} - \left(1 - \frac{1}{\xi}\right)g^{\mu 0}g^{\nu 0} \right]}_{\equiv a^{\mu\nu}\partial_t\partial_t} \partial_t\partial_t \\ &\quad + \underbrace{\left(\frac{1}{\xi} - 1 \right) (g^{\mu 0}g^{\nu k} + g^{\mu k}g^{\nu 0})}_{\equiv b_k^{\mu\nu}\partial_t\partial_k} \partial_t\partial_k + \text{terms without } \partial_t. \end{aligned} \quad (10.39)$$

Application of $\mathcal{D}_x^{\mu\nu}$ to $iD_F^{\mu\nu}(x, y)$:

Note: Apply ∂_t to θ -functions of T ordering as well (product rule!).

$$\begin{aligned} \partial_{x_0} \langle 0|TF(x)G(y)|0\rangle &= \underbrace{\langle 0|(\partial_{x_0}T)F(x)G(y)|0\rangle}_{\substack{= (\partial_{x_0}\theta(x_0 - y_0))\langle 0|F(x)G(y)|0\rangle \\ + (\partial_{x_0}\theta(y_0 - x_0))\langle 0|G(y)F(x)|0\rangle}} + \langle 0|T(\partial_{x_0}F(x))G(y)|0\rangle \\ &= \delta(x_0 - y_0) \langle 0|[F(x), G(y)]|0\rangle \\ &= \delta(x_0 - y_0) \langle 0|[F(x), G(y)]|0\rangle + \langle 0|T\dot{F}(x)G(y)|0\rangle. \end{aligned} \quad (10.40)$$

$$\begin{aligned} \Rightarrow \mathcal{D}_x^{\mu\nu} \langle 0|TA_\nu(x)A_\rho(y)|0\rangle &= \langle 0|T(\underbrace{\mathcal{D}_x^{\mu\nu}A_\nu(x)}_{= 0, \text{ EOM}})A_\rho(y)|0\rangle + b_k^{\mu\nu} \langle 0|(\partial_{x_0}T)(\partial^k A_\nu(x))A_\rho(y)|0\rangle \\ &\quad + 2a^{\mu\nu} \langle 0|(\partial_{x_0}T)(\partial_{x_0}A_\nu(x))A_\rho(y)|0\rangle + a^{\mu\nu} \langle 0|(\partial_{x_0}^2 T)A_\nu(x)A_\rho(y)|0\rangle \\ &= b_k^{\mu\nu} \underbrace{\delta(x_0 - y_0) \langle 0|[\partial^k A_\nu(x), A_\rho(y)]|0\rangle}_{= 0} \\ &\quad + a^{\mu\nu} \langle 0|(\partial_{x_0}T)(\partial_{x_0}A_\nu(x))A_\rho(y)|0\rangle + a^{\mu\nu}\partial_{x_0} \langle 0|(\partial_{x_0}T)A_\nu(x)A_\rho(y)|0\rangle \end{aligned}$$

$$\begin{aligned}
&= a^{\mu\nu} \underbrace{\delta(x_0 - y_0) \langle 0 | [\dot{A}_\nu(x), A_\rho(y)] | 0 \rangle}_{= i[g_{\nu\rho} + g_{\nu 0}g_{\rho 0}(\xi - 1)]\delta(x - y)} + a^{\mu\nu} \partial_{x_0} \underbrace{\left(\delta(x_0 - y_0) \langle 0 | [A_\nu(x), A_\rho(y)] | 0 \rangle \right)}_{= 0} \\
&= \left[g^{\mu\nu} - \left(1 - \frac{1}{\xi} \right) g^{\mu 0} g^{\nu 0} \right] i [g_{\nu\rho} + g_{\nu 0}g_{\rho 0}(\xi - 1)] \delta(x - y) \\
&= i\delta_\rho^\mu \delta(x - y). \tag{10.41}
\end{aligned}$$

Identification of $D_F^{\mu\nu}(x, y)$ as Green function:

$$\left[g^{\mu\nu} \square_x - \left(1 - \frac{1}{\xi} \right) \partial_x^\mu \partial_x^\nu \right] D_{\nu\rho}(x, y) = \delta_\rho^\mu \delta(x - y). \tag{10.42}$$

Solution in momentum space:

$$D_{\nu\rho}(x, y) = \int \frac{d^4k}{(2\pi)^4} e^{-ik(x-y)} \tilde{D}_{\nu\rho}(k). \tag{10.43}$$

$$\hookrightarrow \left[-g^{\mu\nu} k^2 + \left(1 - \frac{1}{\xi} \right) k^\mu k^\nu \right] \tilde{D}_{\nu\rho}(k) = \delta_\rho^\mu$$

$$\tilde{D}_{\nu\rho}(k) = \left[-g_{\nu\rho} k^2 + \left(1 - \frac{1}{\xi} \right) k_\nu k_\rho \right]^{-1} = \frac{-g_{\nu\rho}}{k^2} + \frac{k_\nu k_\rho}{(k^2)^2} (1 - \xi). \tag{10.44}$$

Implementing causal behaviour via Feynman's prescription yields:

$$\begin{aligned}
iD_F^{\mu\nu}(x, y) &= \int \frac{d^4k}{(2\pi)^4} e^{-ik(x-y)} \left[\frac{-ig^{\mu\nu}}{k^2 + i\epsilon} + \frac{ik^\mu k^\nu}{(k^2 + i\epsilon)^2} (1 - \xi) \right] \\
&\stackrel{\xi=1}{=} \int \frac{d^4k}{(2\pi)^4} e^{-ik(x-y)} \frac{-ig^{\mu\nu}}{k^2 + i\epsilon}. \tag{10.45}
\end{aligned}$$

|| Comment:
 || Propagator does not exist for $\xi \rightarrow \infty$, where the gauge-fixing term in \mathcal{L} disappears
 || and the wave operator $\mathcal{D}^{\mu\nu}$ is singular (i.e. contains zero modes = gauge d.o.f.).

Chapter 11

Interacting vector-boson fields

11.1 Electromagnetic interaction

Charged particles and elmg. gauge invariance:

- Charged particles \rightarrow complex fields / non-hermitian field operators Φ, Φ^\dagger ,
otherwise: particle \equiv antiparticle.

Examples: $\Phi, \Phi^\dagger = \phi, \phi^\dagger$ (scalar); $\psi, \bar{\psi}$ (Dirac fermion); V_μ, V_μ^\dagger (vector).

- Charge conservation \rightarrow EOM invariant under *global elmg. gauge transformation*:

$$\Phi \rightarrow \Phi' = e^{-iq\omega} \Phi, \quad \Phi^\dagger \rightarrow \Phi'^\dagger = e^{iq\omega} \Phi^\dagger, \quad \omega = \text{const.} \quad (11.1)$$

- Lagrangian is invariant under global transformation:

$$\mathcal{L}_\Phi(\Phi, \partial\Phi, \dots) = \mathcal{L}_\Phi(\Phi', \partial\Phi', \dots). \quad (11.2)$$

Noether current \rightarrow conserved elmg. current.

“Gauge principle”:

Total Lagrangian should be invariant under *local* gauge transformations:

$$\Phi \rightarrow \Phi' = e^{-iq\omega(x)}\Phi, \quad \text{etc.,} \quad \omega = \omega(x) = \text{arbitrary function.} \quad (11.3)$$

$\hookrightarrow \partial_\mu\Phi$ terms cause problems:

$$\begin{aligned} \partial_\mu\Phi \rightarrow \partial_\mu\Phi' &= \partial_\mu(e^{-iq\omega(x)}\Phi) \\ &= \underbrace{-iq(\partial_\mu\omega)e^{-iq\omega}\Phi}_{\substack{\omega\text{-dependence does} \\ \text{not cancel in } \mathcal{L}_\Phi \\ \text{in general}}} + \underbrace{e^{-iq\omega}(\partial_\mu\Phi)}_{\substack{\omega\text{-dependence cancels} \\ \text{because of global} \\ \text{invariance of } \mathcal{L}_\Phi}}. \end{aligned} \quad (11.4)$$

Idea: replace ∂_μ by *covariant derivative*

$$D_\mu = \partial_\mu + iqA_\mu(x) \quad (11.5)$$

and transform the new field $A_\mu(x)$ such that

$$D_\mu\Phi \rightarrow (D_\mu\Phi)' = D'_\mu\Phi' \stackrel{!}{=} e^{-iq\omega(x)}(D_\mu\Phi). \quad (11.6)$$

Explicitly:

$$D'_\mu\Phi' = (\partial_\mu + iqA'_\mu)(e^{-iq\omega}\Phi) = e^{-iq\omega} \left[\partial_\mu \underbrace{-iq(\partial_\mu\omega) + iqA'_\mu}_{\stackrel{!}{=} iqA_\mu} \right] \Phi. \quad (11.7)$$

$$\Rightarrow A'_\mu = A_\mu + \partial_\mu\omega, \quad \text{elmg. gauge transformation} \quad (11.8)$$

$\hookrightarrow A_\mu(x)$ can be identified with photon field.

Introduction of elmg. interaction by *minimal substitution*:

$$\begin{aligned} \mathcal{L}_\Phi(\Phi, \partial\Phi, \dots) &= \text{globally invariant} \\ \downarrow \partial_\mu \rightarrow D_\mu, \text{ addition of } \mathcal{L}_{A,0} & \\ \mathcal{L} = \mathcal{L}_\Phi(\Phi, D\Phi, \dots) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} &= \text{locally invariant} \end{aligned} \quad (11.9)$$

Examples:

- *Scalar quantum electrodynamics:*

$$\mathcal{L} = (D\phi)^\dagger(D\phi) - m^2\phi^\dagger\phi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \quad D_\mu = \partial_\mu + iqA_\mu, \quad (11.10)$$

ϕ describes a scalar (spin-0) boson with charge q and mass m .

EOMs:

$$[(\partial_\mu + iqA_\mu)(\partial^\mu + iqA^\mu) + m^2]\phi = 0, \quad \text{KG eq. with elmg. interaction, (11.11)}$$

$$\partial_\mu F^{\mu\nu} = j^\nu, \quad \text{Maxwell eq. (11.12)}$$

Elmg. current:

$$\begin{aligned} j^\mu &= \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\frac{\partial\phi}{\partial\omega} + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^\dagger)}\frac{\partial\phi^\dagger}{\partial\omega} \\ &= -iq[(\partial^\mu\phi^\dagger)\phi - \phi^\dagger(\partial^\mu\phi)] - 2q^2A^\mu\phi^\dagger\phi. \end{aligned} \quad (11.13)$$

- *Spinor quantum electrodynamics:*

$$\mathcal{L} = \bar{\psi}(i\not{D} - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \quad D_\mu = \partial_\mu + iqA_\mu, \quad (11.14)$$

ψ describes a Dirac fermion (spin- $\frac{1}{2}$) with charge q and mass m .

EOMs:

$$(i\not{\partial} - q\not{A} - m)\psi = 0, \quad \text{Dirac eq. with elmg. interaction, (11.15)}$$

$$\partial_\mu F^{\mu\nu} = j^\nu, \quad \text{Maxwell eq. (11.16)}$$

Elmg. current: (note fermionic signs !)

$$j^\mu = -\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\frac{\partial\psi}{\partial\omega} + \frac{\partial\bar{\psi}}{\partial\omega}\frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{\psi})} = q\bar{\psi}\gamma^\mu\psi. \quad (11.17)$$

11.2 Perturbation theory for spinor electrodynamics

Lagrangian and quantization: (covariant gauge)

$$\mathcal{L}(A, \psi, \bar{\psi}) = \mathcal{L}_{\psi,0} + \mathcal{L}_{A,0} + \mathcal{L}_{\text{int}}, \quad (11.18)$$

$$\mathcal{L}_{\psi,0}(\psi, \bar{\psi}) = : \bar{\psi} (i\cancel{\partial} - m_f) \psi :, \quad \text{Dirac fermion of mass } m_f \text{ and charge } q = Q_f e,$$

$$\mathcal{L}_{A,0}(A) = -\frac{1}{4} : F_{\mu\nu} F^{\mu\nu} : - \frac{1}{2\xi} : (\partial A)^2 :, \quad \text{photon field,}$$

$$\mathcal{L}_{\text{int}}(A, \psi, \bar{\psi}) = -Q_f e : \bar{\psi} \not{A} \psi : = - : j_\mu A^\mu :, \quad \text{elmg. interaction.}$$

Hamiltonian:

$$\mathcal{H}_{\text{int}}(A, \psi, \bar{\psi}) = -\mathcal{L}_{\text{int}}(A, \psi, \bar{\psi}), \quad \text{since no derivative involved.} \quad (11.19)$$

Quantization of free fields as usual \rightarrow free propagators:

$$\langle 0 | T A^\mu(x) A^\nu(y) | 0 \rangle \Big|_{\text{free}} = i D_F^{\mu\nu}(x, y) \quad \begin{array}{c} \mu \\ \bullet \\ x \end{array} \begin{array}{c} \nu \\ \bullet \\ y \end{array} \quad (11.20)$$

$$\langle 0 | T \psi(x) \bar{\psi}(y) | 0 \rangle \Big|_{\text{free}} = i S_F(x, y) \quad \begin{array}{c} \bullet \\ x \end{array} \begin{array}{c} \bullet \\ y \end{array} \quad (11.21)$$


11.2.1 Expansion of the S-operator

\hookrightarrow Apply Wick theorem as in Yukawa theory (see Sect.9.2):

$$\begin{aligned} S &= T \exp \left[(-iQ_f e) \int d^4x : \bar{\psi}(x) \not{A}(x) \psi(x) : \right] \\ &= \mathbf{1} + (-iQ_f e) \int d^4x : \bar{\psi}(x) \not{A}(x) \psi(x) : + \mathcal{O}(e^2) \end{aligned} \quad (11.22)$$

$$\begin{aligned} &= \mathbf{1} + \begin{array}{c} \diagdown \\ \bullet \\ x \end{array} \begin{array}{c} \diagup \\ \bullet \\ x \end{array} \begin{array}{c} \text{wavy} \\ \bullet \\ x \end{array} + \begin{array}{c} \diagdown \\ \bullet \\ x_1 \end{array} \begin{array}{c} \diagup \\ \bullet \\ x_1 \end{array} \begin{array}{c} \text{wavy} \\ \bullet \\ x_1 \end{array} + \begin{array}{c} \diagdown \\ \bullet \\ x_2 \end{array} \begin{array}{c} \diagup \\ \bullet \\ x_2 \end{array} \begin{array}{c} \text{wavy} \\ \bullet \\ x_2 \end{array} \\ &+ \begin{array}{c} \diagdown \\ \bullet \\ x_1 \end{array} \begin{array}{c} \text{wavy} \\ \bullet \\ x_1 \end{array} \begin{array}{c} \diagup \\ \bullet \\ x_2 \end{array} \begin{array}{c} \diagdown \\ \bullet \\ x_2 \end{array} \begin{array}{c} \diagup \\ \bullet \\ x_2 \end{array} + \begin{array}{c} \diagdown \\ \bullet \\ x_1 \end{array} \begin{array}{c} \text{wavy} \\ \bullet \\ x_1 \end{array} \begin{array}{c} \text{wavy} \\ \bullet \\ x_2 \end{array} \begin{array}{c} \diagup \\ \bullet \\ x_2 \end{array} \begin{array}{c} \diagdown \\ \bullet \\ x_2 \end{array} \begin{array}{c} \text{wavy} \\ \bullet \\ x_2 \end{array} \\ &+ \begin{array}{c} \text{wavy} \\ \bullet \\ x_1 \end{array} \begin{array}{c} \text{wavy} \\ \bullet \\ x_2 \end{array} \begin{array}{c} \text{wavy} \\ \bullet \\ x_1 \end{array} \begin{array}{c} \text{wavy} \\ \bullet \\ x_2 \end{array} + \begin{array}{c} \text{wavy} \\ \bullet \\ x_1 \end{array} \begin{array}{c} \text{wavy} \\ \bullet \\ x_2 \end{array} \begin{array}{c} \text{wavy} \\ \bullet \\ x_1 \end{array} \begin{array}{c} \text{wavy} \\ \bullet \\ x_2 \end{array} + \begin{array}{c} \text{wavy} \\ \bullet \\ x_1 \end{array} \begin{array}{c} \text{wavy} \\ \bullet \\ x_2 \end{array} \begin{array}{c} \text{wavy} \\ \bullet \\ x_1 \end{array} \begin{array}{c} \text{wavy} \\ \bullet \\ x_2 \end{array} \\ &+ \begin{array}{c} \text{wavy} \\ \bullet \\ x_1 \end{array} \begin{array}{c} \text{wavy} \\ \bullet \\ x_2 \end{array} \begin{array}{c} \text{wavy} \\ \bullet \\ x_1 \end{array} \begin{array}{c} \text{wavy} \\ \bullet \\ x_2 \end{array} + \mathcal{O}(e^3) \end{aligned}$$

Feynman rules for graphical representation of the terms $\propto e^n$:

1. Draw all possible diagrams with n vertices  (any number of external lines, including disconnected diagrams).
2. Translate graphs into analytical expressions as follows:
 - External lines $\hat{=}$ non-contracted fields:

$$\begin{aligned}
 A^\mu(x) &= \text{wavy line} \bullet_x \\
 \psi(x) &= \text{arrow} \bullet_x \\
 \bar{\psi}(x) &= \text{arrow} \bullet_x
 \end{aligned}
 \tag{11.23}$$

- Internal lines $\hat{=}$ contracted fields (=propagators):

$$\begin{aligned}
 \overbrace{A^\mu(x_1)A^\nu(x_2)} &= \text{wavy line} \bullet_{x_1}^\mu \bullet_{x_2}^\nu \\
 \overbrace{\psi(x_1)\bar{\psi}(x_2)} &= \text{arrow} \bullet_{x_1} \bullet_{x_2}
 \end{aligned}
 \tag{11.24}$$

- Vertices $\hat{=}$ interaction terms:

$$-iQ_f e \gamma_\mu = \text{vertex diagram}
 \tag{11.25}$$

3. Order terms opposite to the fermion flow indicated by the arrows.
4. For each closed fermion loop take Dirac trace and multiply by (-1) .
5. Integrate the sum of all terms according to

$$\frac{1}{n!} \int d^4x_1 \dots d^4x_n : \dots : .
 \tag{11.26}$$

11.2.2 Feynman rules for S-matrix elements

Consider $n \rightarrow m$ particle process:

$$|i\rangle = a_{A_1}^\dagger \dots a_{A_n}^\dagger |0\rangle, \quad \langle f| = \langle 0| a_{B_m} \dots a_{B_1}, \quad (11.27)$$

where $A_1, \dots, B_m =$ photons or (anti)fermions A, f, \bar{f} .

\hookrightarrow Contributions to $\langle f|S|i\rangle$ only from terms $\propto a_{B_1}^\dagger \dots a_{B_m}^\dagger a_{A_n} \dots a_{A_1}$ in $S!$

Procedure as in Yukawa model, new ingredients for photons:

- External photons: contractions with operators $A^\mu(x)$

$$A^\mu(x) a_\lambda^\dagger(\vec{p}) |0\rangle = \int d\tilde{k} \sum_{\lambda'=1,2} \left[e^{-ikx} \varepsilon_{\lambda'}^\mu(k) \underbrace{a_{\lambda'}(\vec{k}) a_\lambda^\dagger(\vec{p})}_{= a_\lambda^\dagger(\vec{p}) a_{\lambda'}(\vec{k}) + [a_{\lambda'}(\vec{k}), a_\lambda^\dagger(\vec{p})]} + \dots \right] |0\rangle \quad (11.28)$$

$$= e^{-ipx} \varepsilon_\lambda^\mu(p) |0\rangle + \dots \quad (11.29)$$

\hookrightarrow Contractions with incoming /outgoing photons:

$$\begin{aligned} \overbrace{A^\mu(x) a_\lambda^\dagger(\vec{p})} |0\rangle &= e^{-ipx} \varepsilon_\lambda^\mu(p) |0\rangle, \\ \langle 0| \overbrace{a_\lambda(\vec{p}) A^\mu(x)} &= \langle 0| e^{ipx} \varepsilon_\lambda^\mu(p)^*. \end{aligned} \quad (11.30)$$

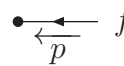
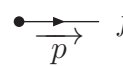
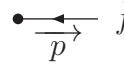
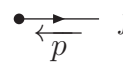
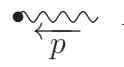
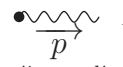
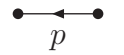
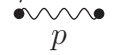
- Internal photons: Fourier representation of propagator

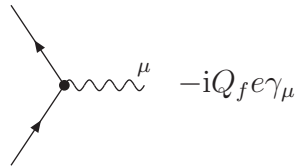
$$\overbrace{A^\mu(x_1) A^\nu(x_2)} = iD_F^{\mu\nu}(x_1, x_2) = \int \frac{d^4k}{(2\pi)^4} e^{-ik(x_1-x_2)} \left[\frac{-ig^{\mu\nu}}{k^2 + i\epsilon} + \frac{ik^\mu k^\nu}{(k^2 + i\epsilon)^2} (1 - \xi) \right]. \quad (11.31)$$

- Space-time integrals $\int d^4x_i$ at vertices imply momentum conservation,
loop integrals $\int d^4p_l$ remain open, fermionic signs as usual ...

Feynman rules for the transition matrix element \mathcal{M}_{fi} :

1. Determine all relevant Feynman diagrams:
 - $n \rightarrow m$ scattering process $\Rightarrow n + m$ external lines.
 - Order of perturbation theory \Rightarrow number of loops.
2. Impose momentum conservation at each vertex.
3. Insert the explicit expressions (fermionic terms ordered opposite to arrows):

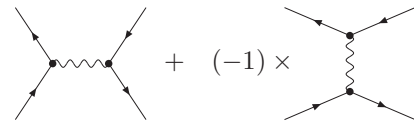
 $u_\sigma(p)$	 $\bar{u}_\sigma(p)$
 $v_\sigma(p)$	 $\bar{v}_\sigma(p)$
 $\varepsilon^\mu(p)$	 $\varepsilon^\mu(p)^*$
 $\frac{i}{\not{p} - m_f + i\epsilon}$	 $\frac{-ig^{\mu\nu}}{p^2 + i\epsilon} + \frac{ip^\mu p^\nu}{(p^2 + i\epsilon)^2}(1 - \xi)$


 $-iQ_f e \gamma_\mu$

(11.32)

4. Integrate over all loop momenta p_l via $\int \frac{d^4 p_l}{(2\pi)^4}$.
5. For each closed fermion loop take Dirac trace and multiply by (-1) . Insert a relative sign between diagrams that result from interchanging external fermion lines.

Example:


 $+ (-1) \times$

6. The coherent sum of all diagrams yields $i\mathcal{M}_{fi}$.

Straightforward generalization to more charged fermions f :

Propagation of free fields completely independent,

interaction: $\mathcal{L}_{\text{int}} = -\sum_f Q_f e : \bar{\psi}_f A \psi_f : .$

\Rightarrow Each fermion f has its own propagator and $\gamma^f f \bar{f}$ vertex.

Comments:

- Gauge-parameter independence of S -matrix elements:

Momentum terms $\propto p^\mu p^\nu$ in photon propagator do not contribute to amplitudes.

\Leftrightarrow ξ -dependence completely cancels (proof non-trivial).

- Gauge freedom of external states:

Amplitudes \mathcal{M} with an incoming/outgoing photon of momentum p obey the following (non-trivial) *Ward identity*:

$$\mathcal{M} = \varepsilon_\lambda^\mu(p)^{(*)} T_\mu(p) \quad \Rightarrow \quad p^\mu T_\mu(p) = 0, \quad (11.33)$$

where all external states other than the photon must be on shell.

\Rightarrow Physical (transverse) polarization vectors can be changed according to

$$\varepsilon_{1,2}^\mu(p) \rightarrow \hat{\varepsilon}_{1,2}^\mu(p) = \varepsilon_{1,2}^\mu(p) + a p^\mu \quad \text{with } a = \text{arbitrary}. \quad (11.34)$$

|| Comment:
|| A Lorentz transformation in general leads to such replacements.

- Convenient construction of transverse polarizations:

Choose any *gauge vector* $n^\mu = (1, \vec{n})$ with $\vec{n}^2 = 1$, $n^2 = 0$, $p_\mu n^\mu \neq 0$ and define $\varepsilon_{1,2}^\mu(p)$ such that

$$p_\mu \varepsilon_{1,2}^\mu(p) = n_\mu \varepsilon_{1,2}^\mu(p) = 0. \quad (11.35)$$

|| Comment:
|| This is possible due to the gauge freedom (11.34).

\Rightarrow Completeness relation:

$$\sum_{\lambda=1,2} \varepsilon_\lambda^\mu(p) \varepsilon_\lambda^\nu(p)^* = -g^{\mu\nu} + \underbrace{\frac{p^\mu n^\nu + p^\nu n^\mu}{p \cdot n}}. \quad (11.36)$$

does not contribute to amplitudes
because of Ward identity (11.33)

\Leftrightarrow Convenient in photon spin summation of squared amplitudes.

- Unphysical parts $|\psi'\rangle = c_0 |0\rangle + |N, 1\rangle + \dots$ of photon states $|\psi\rangle = |\psi_T\rangle \otimes |\psi'\rangle$ do not influence transition amplitudes, because S is derived from the exponential of the Hamiltonian $H = H_T \oplus H'$, so that $S = S_T \otimes S'$. Schematically:

$$\frac{\langle f|S|i\rangle}{\|f\| \cdot \|i\|} = \frac{\langle f_T|S_T|i_T\rangle}{\|f_T\| \cdot \|i_T\|} \underbrace{\left(\frac{\langle f'|S'|i'\rangle}{\|f'\| \cdot \|i'\|} \right)}. \quad (11.37)$$

can only be a phase factor,
because $\langle f'|S'|i'\rangle = c_{0,f}^* c_{0,i} \langle 0|0\rangle$,
see Sect. 10.3.2

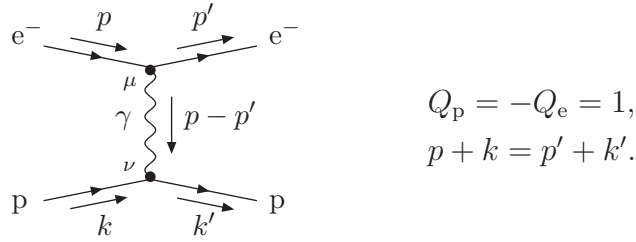
11.3 Important processes of (spinor) QED

11.3.1 Elastic ep scattering

Process:

$$e^-(p, \sigma) + p(k, \tau) \rightarrow e^-(p', \sigma') + p(k', \tau') \quad (11.38)$$

Diagram:



Amplitude in Born approximation (tree level):

$$\begin{aligned} i\mathcal{M} &= \left[\bar{u}_{\sigma'}(p') (-iQ_e e) \gamma_\mu u_\sigma(p) \right] \left[\bar{u}_{\tau'}(k') (-iQ_p e) \gamma_\nu u_\tau(k) \right] \\ &\times \left[\underbrace{\frac{-ig^{\mu\nu}}{(p-p')^2 + i\epsilon}}_{\substack{\text{ie irrelevant} \\ \text{at Born level}}} + \underbrace{\frac{i(p-p')^\mu(p-p')^\nu}{((p-p')^2 + i\epsilon)^2} (1-\xi)}_{\substack{\text{no contribution due to Dirac eqs.:} \\ \bar{u}_{\sigma'}(p')(\not{p}-\not{p}')u_\sigma(p) = 0}} \right] \\ &= \frac{iQ_e Q_p e^2}{(p-p')^2} \left[\bar{u}_{\sigma'}(p') \gamma_\mu u_\sigma(p) \right] \left[\bar{u}_{\tau'}(k') \gamma^\mu u_\tau(k) \right]. \end{aligned} \quad (11.39)$$

$$\begin{aligned} \Rightarrow \overline{|\mathcal{M}|^2} &= \frac{1}{4} \sum_{\sigma, \sigma', \tau, \tau'} |\mathcal{M}|^2 \\ &= \frac{Q_e^2 Q_p^2 e^4}{4(p-p')^4} \text{Tr}\{(\not{p}' + m_e)\gamma_\mu(\not{p} + m_e)\gamma_\nu\} \text{Tr}\{(\not{k}' + m_p)\gamma^\mu(\not{k} + m_p)\gamma^\nu\} \\ &= \frac{4e^4}{(p-p')^4} \{p'_\mu p_\nu + p'_\nu p_\mu - (pp' - m_e^2)g_{\mu\nu}\} \{k'^\mu k^\nu + k'^\nu k^\mu - (kk' - m_p^2)g^{\mu\nu}\} \\ &= \dots = \frac{2e^4}{t^2} \left\{ t^2 + 2st + 2(s - m_e^2 - m_p^2)^2 \right\}, \end{aligned} \quad (11.40)$$

$$\text{where } s = (p+k)^2, \quad t = (p-p')^2, \quad u = (p-k')^2. \quad (11.41)$$

Kinematics in proton rest frame: (xz -plane = scattering plane)

$$\begin{aligned} p^\mu &= (E_e, 0, 0, p_e), & p_e &= \sqrt{E_e^2 - m_e^2}, \\ p'^\mu &= (E'_e, p'_e \sin \theta, 0, p'_e \cos \theta), & p'_e &= \sqrt{E_e'^2 - m_e^2}, \\ k^\mu &= (m_p, 0, 0, 0). \end{aligned} \quad (11.42)$$

$$\Rightarrow s = m_e^2 + m_p^2 + 2pk = m_e^2 + m_p^2 + 2m_p E_e, \quad \sqrt{\lambda(s, m_e^2, m_p^2)} = \dots = 2m_p p_e. \quad (11.43)$$

Note: $E'_e = E'_e(E_e, \theta)$ with $E_e = E'_e(E_e, 0)$.
 E_e given by experiment, measured: E'_e or θ .

Relation between E_e, E'_e, θ from $k'^2 = m_p^2$, e.g. derived via t :

$$\begin{aligned} t &= 2m_e^2 - 2pp' = 2(m_e^2 - E_e E'_e + p_e p'_e \cos \theta) \\ &= (k - k')^2 = 2m_p^2 - 2kk' = 2m_p^2 - 2k(p + k - p') = -2k(p - p') \\ &= -2m_p(E_e - E'_e). \end{aligned} \quad (11.44)$$

$$\Rightarrow E_e - E'_e = -\frac{m_e^2 - E_e E'_e + p_e p'_e \cos \theta}{m_p}, \quad \text{can be solved for } E'_e \quad (11.45)$$

or, e.g., expanded for $m_p \rightarrow \infty$

$$\begin{aligned} &= -\frac{m_e^2 - E_e^2 + p_e^2 \cos \theta}{m_p} + \mathcal{O}(1/m_p^2) \\ &= \frac{p_e^2(1 - \cos \theta)}{m_p} + \dots \end{aligned} \quad (11.46)$$

2-particle phase space:

$$\int d\Phi_2 = \underbrace{\int d\varphi_1}_{\substack{\text{azimuthal angle} \\ \text{integral} \rightarrow 2\pi}} \int dt \frac{1}{4(2\pi)^2} \frac{1}{\sqrt{\lambda(s, m_e^2, m_p^2)}}, \quad \text{derived as in Exercise 7.1.} \quad (11.47)$$

$\lambda(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2xz - 2yz$

Cross section:

$$\int d\sigma = \underbrace{\frac{1}{2\sqrt{\lambda(s, m_e^2, m_p^2)}}}_{\text{flux factor}} \int d\Phi_2 |\overline{\mathcal{M}}|^2. \quad (11.48)$$

Differential cross section in *virtuality* $Q^2 \equiv -t$ of the photon:

$$\begin{aligned} \frac{d\sigma}{dQ^2} &= \frac{1}{16\pi} \frac{1}{\lambda(s, m_e^2, m_p^2)} \frac{2e^4}{t^2} \left\{ t^2 + 2st + 2(s - m_e^2 - m_p^2)^2 \right\} \\ &= \frac{\pi\alpha^2}{2m_p^2 p_e^2} \left\{ \frac{8m_p^2 E_e^2}{Q^4} - \frac{2(m_e^2 + m_p^2 + 2m_p E_e)}{Q^2} + 1 \right\}. \end{aligned} \quad (11.49)$$

= Lorentz invariant, since $d\sigma$ and Q^2 are invariant,

$$\text{where } \alpha = \frac{e^2}{4\pi} = \frac{e^2}{4\pi\epsilon_0\hbar c} = 1/137.0\dots = \text{fine-structure constant.} \quad (11.50)$$

Interesting kinematical limits:

- Non-relativistic limit: $p_e, p'_e \ll m_e \ll m_p$

Use approximation (11.46) of large m_p for Q^2 :

$$\begin{aligned} Q^2 &= -t = 2p_e^2(1 - \cos \theta) + \mathcal{O}(1/m_p) = 4p_e^2 \sin^2 \frac{\theta}{2} + \dots, \\ dQ^2 &= -2p_e^2 d \cos \theta + \dots \end{aligned} \quad (11.51)$$

and neglect all terms of $\mathcal{O}\left(\frac{E_e^{(\prime)}}{m_p} \sim \frac{m_e}{m_p}, \frac{p_e^{(\prime)}}{m_e}\right)$ relative to the leading term.

↔ Differential cross section in angle θ :

$$\frac{d\sigma}{d \cos \theta} = \frac{\pi\alpha^2}{m_p^2} \frac{8m_p^2 m_e^2}{Q^4} + \dots = \frac{\pi\alpha^2 m_e^2}{2p_e^4 \sin^4 \frac{\theta}{2}} + \dots, \quad \text{Rutherford cross section.} \quad (11.52)$$

Comment: The scattering is determined by the particle charges only, the spin does not play a role in the non-relativistic limit.

- Relativistic electron: $m_e \ll p_e, p'_e \ll m_p$

Again use expansion (11.51) for Q^2 and neglect all m_e terms (i.e. $p_e \rightarrow E_e$).

↔ Differential cross section in angle θ :

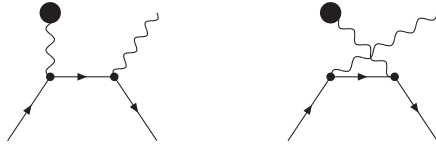
$$\frac{d\sigma}{d \cos \theta} = \frac{\pi\alpha^2}{m_p^2} \left\{ \frac{8m_p^2 E_e^2}{Q^4} - \frac{2m_p^2}{Q^2} \right\} + \dots = \frac{\pi\alpha^2 \cos^2 \frac{\theta}{2}}{2E_e^2 \sin^4 \frac{\theta}{2}} + \dots, \quad \text{Mott cross section.} \quad (11.53)$$

Comments:

- Electron helicity is conserved in the relativistic scattering process. The factor $\cos^2 \frac{\theta}{2}$ results from e^- spin rotation.
- For E_e above ~ 100 MeV ($m_p \sim 1$ GeV) the finite extension of the proton becomes visible.
 - ↔ *Formfactor* for proton charge distribution necessary.
- For Q^2 above $\sim m_p^2$ inelastic scattering becomes relevant (proton breaks up).
 - ↔ Measured in terms of *structure functions*, described by the *parton model*.

11.3.2 Other important processes in QED

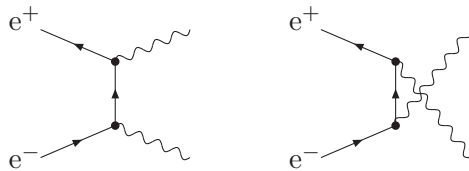
- *Bremsstrahlung* in potential scattering: $e^- + A \rightarrow e^- + A + \gamma$



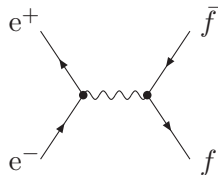
- *Compton scattering*: $e^- + \gamma \rightarrow e^- + \gamma$



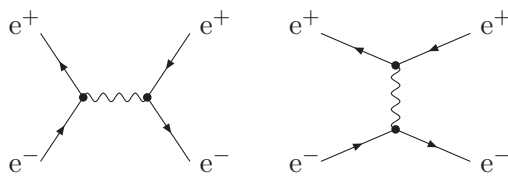
- *Pair annihilation*: $e^+ + e^- \rightarrow \gamma\gamma$



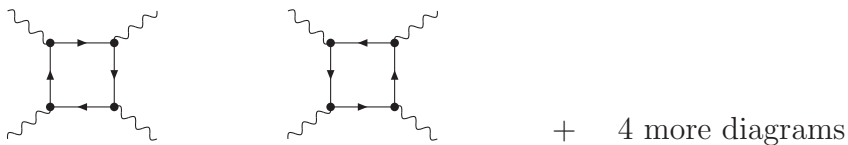
- *Pair creation*: $e^+ + e^- \rightarrow f\bar{f}$



- *Bhabha scattering*: $e^+ + e^- \rightarrow e^+ + e^-$



- *$\gamma\gamma$ scattering*: $\gamma\gamma \rightarrow \gamma\gamma$



Note:

Higher-order processes of QED violate the superposition principle for elmg. fields !

- etc.

Bibliography

- [1] J. D. Bjorken and S. D. Drell, “Relativistic Quantum Mechanics”, McGraw-Hill (1964), German translation BI-Wissenschaftsverlag (1990)
- [2] J. D. Bjorken and S. D. Drell, “Relativistic Quantum Fields”, McGraw-Hill (1965), Reprint: Dover (2012). German translation BI-Wissenschaftsverlag (1990)
- [3] C. Itzykson and J. B. Zuber, “Quantum Field Theory,” New York, Usa: McGraw-hill (1980). Reprint: Dover (2006)
- [4] M. Maggiore, “A Modern introduction to quantum field theory,” Oxford University Press (2005)
- [5] M. E. Peskin and D. V. Schroeder, “An Introduction to quantum field theory,” Reading, USA: Addison-Wesley (1995) 842 p
- [6] S. Weinberg, “The Quantum theory of fields. Vol. 1: Foundations,” Cambridge, UK: Univ. Pr. (1995) 609 p
- [7] M. D. Schwartz, “Quantum Field Theory and the Standard Model,” Cambridge University Press (2013), 859 p
- [8] M. Srednicki, “Quantum field theory,” Cambridge, UK: Univ. Pr. (2007) 641 p
The parts “Spin zero” and “Spin one-half” are also available at
<http://arxiv.org/abs/hep-th/0409035> and
<http://arxiv.org/abs/hep-th/0409036>
- [9] H. Goldstein, C. Poole, J. Safko, “Classical Mechanics,” Addison Wesley (2002).