Exercise 8.1   *S-operator for two interacting scalar fields (cont'd)*   (1 point)

Consider again the field theory of a complex scalar field ϕ (particle ϕ and antiparticle $\bar{\phi}$) and a real scalar field Φ (particle Φ) from Exercise 7.2 with the Lagrangian density

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \Phi)(\partial^\mu \Phi) - \frac{1}{2}M^2 \Phi^2 + (\partial_\mu \phi^\dagger)(\partial^\mu \phi) - m^2 \phi^\dagger \phi + \mathcal{L}_{\text{int}},$$

where $\mathcal{L}_{\text{int}} = \lambda \phi^\dagger \phi\, \Phi$, and make use of the perturbative expansion worked out there.

a) Calculate the S-matrix element $S_{fi} = \langle f|S|i \rangle$ in lowest order between the initial state $|i\rangle = a_\Phi^\dagger(k)|0\rangle$ and the final state $|f\rangle = a_\phi^\dagger(p_1)\, b_\phi^\dagger(p_2)|0\rangle$, where $a_\Phi^\dagger(q)$, $a_\phi^\dagger(q)$, $b_\phi^\dagger(q)$ are the creation operators of the particles Φ, ϕ, and $\bar{\phi}$.

b) Assuming $M > 2m$, calculate the lowest-order decay width

$$\Gamma_{\Phi \to \phi\bar{\phi}} = \frac{1}{2M} \int d\Phi_2\, |\mathcal{M}_{fi}|^2$$

for the decay $\Phi \to \phi\bar{\phi}$, where Φ_2 is the 2-particle phase space of the final state (see Exercise 5.2) and the transition matrix element $\mathcal{M}_{fi}$ is related to S_{fi} by

$$S_{fi} = (2\pi)^4 \delta(k - p_1 - p_2)\, i\mathcal{M}_{fi}.$$

Exercise 8.2   *Fundamental representations of the Lorentz group*   (1 point)

The general form of Lorentz transformations in the two fundamental representations of the Lorentz group is given by

$$\Lambda_{\text{R}} = \exp\left(-\frac{i}{2}\sum_{k=1}^3 (\phi_k + i\nu_k)\sigma^k\right), \qquad \Lambda_{\text{L}} = \exp\left(-\frac{i}{2}\sum_{k=1}^3 (\phi_k - i\nu_k)\sigma^k\right)$$

with the real group parameters ϕ_k, ν_k and the Pauli matrices σ^k.

a) Show that $\Lambda_{\text{R}}^\dagger = \Lambda_{\text{L}}^{-1}$ and $\Lambda_{\text{L}}^\dagger = \Lambda_{\text{R}}^{-1}$.

b) Show that $\det(\Lambda_{\text{R}}) = \det(\Lambda_{\text{L}}) = 1$ using $\det(\exp\{A\}) = \exp\{\text{Tr}(A)\}$ for a matrix A.

c) Which transformations are characterized by $\Lambda_{\text{R/L}}^\dagger = \Lambda_{\text{R/L}}$, which by $\Lambda_{\text{R/L}}^\dagger = \Lambda_{\text{R/L}}^{-1}$?

d) Derive Λ_{R} and Λ_{L} for a pure boost in the direction $\vec{e} = \begin{pmatrix} \cos\varphi \sin\theta \\ \sin\varphi \sin\theta \\ \cos\theta \end{pmatrix}$ with $\vec{\nu} = \nu\vec{e}$, $\vec{\phi} = 0$ and for a pure rotation around the axis $\vec{e}$ with $\vec{\phi} = \phi\vec{e}$, $\vec{\nu} = 0$.

Please turn over!

Exercise 8.3 *Connection between Λ_R, Λ_L, and $\Lambda^\mu{}_\nu$* (1 point)

The general matrix representing a Lorentz transformation of a four-vector is given by

$$\Lambda^\mu{}_\nu = \exp\left(-\frac{i}{2}\omega_{\alpha\beta}M^{\alpha\beta}\right)^\mu{}_\nu, \qquad (M^{\alpha\beta})^\mu{}_\nu = i(g^{\alpha\mu}g^\beta{}_\nu - g^{\beta\mu}g^\alpha{}_\nu)$$

with the antisymmetric parameters $\omega_{jk} = \epsilon_{jkl}\phi_l$ and $\omega_{0j} = -\omega_{j0} = \nu_j$. The connection between Λ_R, Λ_L (see Exercise 8.1) and $\Lambda^\mu{}_\nu$ is

$$\Lambda_\mathrm{R}^\dagger \sigma^\mu \Lambda_\mathrm{R} = \Lambda^\mu{}_\nu \sigma^\nu, \qquad \Lambda_\mathrm{L}^\dagger \bar\sigma^\mu \Lambda_\mathrm{L} = \Lambda^\mu{}_\nu \bar\sigma^\nu,$$

where $\sigma^\mu = (\mathbb{1}, \sigma^1, \sigma^2, \sigma^3)$ and $\bar\sigma^\mu = (\mathbb{1}, -\sigma^1, -\sigma^2, -\sigma^3)$. Verify these relations for infinitesimal transformations with the parameters $\delta\phi_k$, $\delta\nu_k$.