Exercises to Group Theory for Physicists — Sheet 4
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Exercise 4.1 *Real, pseudoreal, and complex representations* (6 points)

In the lecture we showed that if a unitary, irreducible representation D of a finite group G acting on a vector space V is real or pseudoreal, i.e. $\exists S$ with $S^{\mathrm{T}} = \pm S$ so that $D^*(g) = SD(g)S^{-1}\,\forall\, g \in G$, then $x^{\mathrm{T}}Sy$, $x, y \in V$, is a bilinear invariant.

a) Show that, conversely, if a bilinear invariant $x^{\mathrm{T}}Sy$ exists with some non-vanishing matrix S, then D must be real or pseudoreal, i.e. that for complex D no such invariant exists.

b) Show that for an arbitrary square matrix X of the same dimension as D and

$$S = \sum_g D(g)^{\mathrm{T}} X D(g), \tag{1}$$

$x^{\mathrm{T}}Sy$ is invariant under group action. What happens if the representation D is complex?

c) Choose the matrix X in (1) as X^{mn} with components $(X^{mn})^{jk} = \delta^{mj}\delta^{nk}$. Use the resulting equation to calculate $\sum_g \chi(g^2)$, where $\chi(g^2)$ is the character of $D(g^2)$, in the case where representation D is complex.

d) Proceed as in c), but in the case where D is a real or pseudoreal representation, i.e. $S^{\mathrm{T}} = \eta S$ with $\eta = +1$ rsp. $\eta = -1$ if the representation is real rsp. pseudoreal. To what does $\sum_g \chi(g^2)$ evaluate in these cases?

e) Check if the 2-dimensional irreducible representations of the quaternionic group Q and of the symmetric group S_3 are real, pseudoreal, or complex (use the character tables given in the lecture).

f) With the number of square roots σ_g of an element $g \in G$, i.e. the number of elements $f \in G : f^2 = g$, we can write

$$\sum_g \chi(g^2) = \sum_g \sigma_g \chi(g). \tag{2}$$

Use this and the results from c) and d) to derive a formula to calculate σ_g.

Hint: Use the character completeness relation.

Please turn over!

Exercise 4.2 *Irreducible representations of the dihedral groups D_n* (3 points)

According to Exercise 2.2, the 2×2 rotation matrices

$$r_k = \begin{pmatrix} \cos\phi_k & -\sin\phi_k \\ \sin\phi_k & \cos\phi_k \end{pmatrix}, \qquad \phi_k = \frac{2\pi k}{n}, \qquad k = 0, 1, \dots, n-1, \tag{3}$$

generate n reducible representations of the cyclic group $C_n = \langle\, r \,|\, r^n = e \,\rangle$. The similarity transformation $S = \frac{1}{\sqrt{2}} \left(\begin{smallmatrix} 1 & \mathrm{i} \\ \mathrm{i} & 1 \end{smallmatrix}\right)$ reduces the generating elements to

$$S \begin{pmatrix} \cos\phi_k & -\sin\phi_k \\ \sin\phi_k & \cos\phi_k \end{pmatrix} S^{-1} = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\phi_k} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\phi_k} \end{pmatrix}. \tag{4}$$

a) As in Exercise 3.1, extend the group C_n by a second generating element p, so that

$$D_n = \langle\, r, p \,|\, r^n = p^2 = e, prp^{-1} = r^{-1} \rangle, \tag{5}$$

but this time for all n generators r_k given in (3), yielding n two-dimensional representations. Apply the similarity transformation S to p. For which of the r_k are the generated representations irreducible? Distinguish the cases of n even and n odd.

b) Which of the 2-dimensional representations are inequivalent?

Hint: It is sufficient to restrict the similarity transformations to those that leave the simultaneously diagonalisable matrices diagonal (why?).

c) Show that, together with the 1-dimensional irreducible representations found in Exercise 3.1, these are all irreducible representations of D_n.