Exercise 9.1 *Relations for Dirac matrices* (2 points)

The Dirac matrices γ_μ are defined by

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}, \qquad \gamma_0\gamma^\mu\gamma_0 = (\gamma^\mu)^\dagger, \qquad \gamma_5 = \gamma^5 = \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3 = -\frac{i}{4!}\epsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma.$$

a) Show the relations

$$\{\gamma^\mu, \gamma^5\} = 0, \qquad (\gamma_5)^\dagger = \gamma_5.$$

Calculate the explicit form of γ_5 in the chiral representation of the Dirac matrices. What is the meaning of the matrices $\omega_\pm = \frac{1}{2}(\mathbf{1} \pm \gamma_5)$?

b) Derive the following traces:

$$\mathrm{Tr}[\gamma^\mu\gamma^\nu], \qquad \mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma].$$

c) Prove the following trace relations:

$$\mathrm{Tr}[\gamma_5] = \mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma_5] = 0, \qquad \mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_5] = -4i\epsilon^{\mu\nu\rho\sigma},$$
$$\mathrm{Tr}[\gamma^{\mu_1}\dots\gamma^{\mu_{2n+1}}] = \mathrm{Tr}[\gamma^{\mu_1}\dots\gamma^{\mu_{2n+1}}\gamma_5] = 0, \quad n = 0, 1, \dots.$$

d) Reduce the number of Dirac matrices in the following contractions:

$$\gamma^\alpha\gamma_\alpha, \qquad \gamma^\alpha\gamma^\mu\gamma_\alpha, \qquad \gamma^\alpha\gamma^\mu\gamma^\nu\gamma_\alpha.$$

Exercise 9.2 *Lorentz covariants from Dirac spinors* (1.5 points)

a) Prove the following relations:

$$S(\Lambda)^{-1}\gamma^\mu S(\Lambda) = \Lambda^\mu{}_\nu\,\gamma^\nu, \qquad S(\Lambda)^\dagger\gamma_0 = \gamma_0 S(\Lambda)^{-1}.$$

b) Using a), show that the quantities

$$\begin{aligned}
s(x) &= \overline{\psi}(x)\psi(x) = \text{scalar},\\
p(x) &= \overline{\psi}(x)\gamma_5\psi(x) = \text{pseudo-scalar},\\
j^\mu(x) &= \overline{\psi}(x)\gamma^\mu\psi(x) = \text{vector},\\
j_5^\mu(x) &= \overline{\psi}(x)\gamma^\mu\gamma_5\psi(x) = \text{pseudo-vector},\\
T^{\mu\nu}(x) &= \overline{\psi}(x)\gamma^\mu\gamma^\nu\psi(x) = \text{rank-2 tensor},\\
T_5^{\mu\nu}(x) &= \overline{\psi}(x)\gamma^\mu\gamma^\nu\gamma_5\psi(x) = \text{rank-2 pseudo-tensor}.
\end{aligned}$$

transform under proper, orthochronous Lorentz transformations $x'^\mu = \Lambda^\mu{}_\nu x^\nu$ as indicated, if the Dirac spinor $\psi(x)$ transforms according to $\psi(x) \to \psi'(x') = S(\Lambda)\psi(x)$.

c) Determine the transformation properties of the quantities defined in b) under the parity operation P, where

$$x'^\mu = (x^0, -\mathbf{x}) = (\Lambda_P)^\mu{}_\nu x^\nu, \qquad S(\Lambda_P) = \gamma_0 = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}$$

in the chiral representation of the Dirac matrices.