Exercise 8.1 *Fundamental representations of the Lorentz group* (1 point)

The general form of Lorentz transformations in the two fundamental representations of the Lorentz group is given by

$$\Lambda_{\mathrm{R}} = \exp\left\{-\frac{\mathrm{i}}{2}(\phi_k + i\nu_k)\sigma^k\right\}, \qquad \Lambda_{\mathrm{L}} = \exp\left\{-\frac{\mathrm{i}}{2}(\phi_k - i\nu_k)\sigma^k\right\}$$

with the real group parameters ϕ_k, ν_k and the Pauli matrices σ^k.

a) Show that $\Lambda_{\mathrm{R}}^\dagger = \Lambda_{\mathrm{L}}^{-1}$ and $\Lambda_{\mathrm{L}}^\dagger = \Lambda_{\mathrm{R}}^{-1}$.

b) Show that $\det(\Lambda_{\mathrm{R}}) = \det(\Lambda_{\mathrm{L}}) = 1$ using $\det(\exp\{A\}) = \exp\{\mathrm{Tr}(A)\}$ for a matrix A.

c) Which transformations are characterized by $\Lambda_{\mathrm{R/L}}^\dagger = \Lambda_{\mathrm{R/L}}$, which by $\Lambda_{\mathrm{R/L}}^\dagger = \Lambda_{\mathrm{R/L}}^{-1}$?

d) Derive Λ_{R} and Λ_{L} for a pure boost in the direction $\mathbf{e} = \begin{pmatrix} \cos\varphi\sin\theta \\ \sin\varphi\sin\theta \\ \cos\theta \end{pmatrix}$ with $\boldsymbol{\nu} = \nu\mathbf{e}$, $\boldsymbol{\phi} = 0$ and for a pure rotation about the axis $\mathbf{e}$ with $\boldsymbol{\phi} = \phi\mathbf{e}$, $\boldsymbol{\nu} = 0$.

Exercise 8.2 *Relation between the Lorentz group and $SL(2,\mathbf{C})$* (1 point)

The group $SL(2,\mathbf{C})$ consists of all complex 2×2 matrices A with $\det(A) = 1$. Assign to each 4-vector x^μ a 2×2 matrix $X = x_\mu\sigma^\mu$ and $\bar{X} = x_\mu\bar{\sigma}^\mu$ where $\sigma^\mu = (\mathbf{1}, \sigma^1, \sigma^2, \sigma^3)$ and $\bar{\sigma}^\mu = (\mathbf{1}, -\sigma^1, -\sigma^2, -\sigma^3)$. The matrices σ^μ and $\bar{\sigma}^\mu$ satisfy the relation $\mathrm{Tr}(\sigma^\mu\bar{\sigma}^\nu) = 2g^{\mu\nu}$.

a) Show that the inverse of the above assignment is given by

$$x^\mu = \frac{1}{2}\mathrm{Tr}(X\bar{\sigma}^\mu) = \frac{1}{2}\mathrm{Tr}(\bar{X}\sigma^\mu) \tag{1}$$

b) What is the meaning of $\det(X)$ and $\det(\bar{X})$?

c) For two arbitrary matrices A, B of $SL(2,\mathbf{C})$, show that the mappings $X \to X' = AXA^\dagger$ and $\bar{X} \to \bar{X}' = B\bar{X}B^\dagger$ define Lorentz transformations $x'^\mu = \Lambda^\mu{}_\nu x^\nu$.
(Hint: Consider the determinants.)

d) How are the matrices A, B of Exercise 8.2 c) and the matrices Λ_{L} and Λ_{R} of the fundamental representations of Exercise 8.1 related?
(Hint: $\Lambda_{\mathrm{R}}^\dagger\sigma^\mu\Lambda_{\mathrm{R}} = \Lambda^\mu{}_\nu\sigma^\nu, \Lambda_{\mathrm{L}}^\dagger\bar{\sigma}^\mu\Lambda_{\mathrm{L}} = \Lambda^\mu{}_\nu\bar{\sigma}^\nu$.)

Please turn over!

Exercise 8.3 *Connection between* $\Lambda_{\rm R}$, $\Lambda_{\rm L}$, *and* $\Lambda^\mu{}_\nu$ (1 point)

The general matrix representing a Lorentz transformation of a four-vector is given by

$$\Lambda^\mu{}_\nu = \exp\left\{-\frac{i}{2}\omega_{\alpha\beta}M^{\alpha\beta}\right\}^\mu{}_\nu, \qquad (M^{\alpha\beta})^\mu{}_\nu = i(g^{\alpha\mu}\delta^\beta_\nu - g^{\beta\mu}\delta^\alpha_\nu)$$

with the antisymmetric parameters $\omega_{jk} = \epsilon_{jkl}\phi_l$ and $\omega_{0j} = -\omega_{j0} = \nu_j$. The connection between $\Lambda_{\rm R}$, $\Lambda_{\rm L}$ (see Exercise 8.1) and $\Lambda^\mu{}_\nu$ is

$$\Lambda_{\rm R}^\dagger\sigma^\mu\Lambda_{\rm R} = \Lambda^\mu{}_\nu\sigma^\nu, \qquad \Lambda_{\rm L}^\dagger\bar\sigma^\mu\Lambda_{\rm L} = \Lambda^\mu{}_\nu\bar\sigma^\nu,$$

where $\sigma^\mu = (\mathbf{1}, \sigma^1, \sigma^2, \sigma^3)$ and $\bar\sigma^\mu = (\mathbf{1}, -\sigma^1, -\sigma^2, -\sigma^3)$. Verify these relations for infinitesimal transformations with the parameters $\delta\phi_k$, $\delta\nu_k$.